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JEMA52: COMBINATORIAL MATHEMATICS

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Unit II

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TEXT BOOK

Issac and Dr. Arumugam S, Sequences and Series and Trigonometry (2014), New Gamma Publishing house.

JEMA52: COMBINATORIAL MATHEMATICS

CONTENTS

Unit I

Selections and binomial coefficients- Permutations-Ordered selections-unordered selections-Miscellaneous problems

Chapter 1: Sections 1.1 to 1.4

1. Selections and binomial coefficients:

1.1. Permutations:

This chapter will investigate the problem of finding how many ways there are of selecting k objects from a set of n objects. There are essentially four different problems here, depending on whether or not selections are ordered, and also on whether or not repetitions are allowed (i.e. whether or not an object can be selected more than once).

To start with, consider the problem of listing in order all the elements of a set of size n. If $p(n)$ denotes the number of such listings, then

$$p(1) = 1, p(2) = 2, p(3) = 6.$$

To see that $p(3) = 6$, let the objects be a, b, c. Then the 6 possible listings are

$$abc, acb, bac, bca, cab, cba$$

Any such ordering is called a permutation of a, b, c.

The general formula for $p(n)$ is now obtained. Choose one of the n objects to be placed first in the list. This can be done in n ways, and each of these n choices results in ($n - 1$) objects being left. These ($n - 1$) objects can be placed in the ($n - 1$) remaining places in $p(n - 1)$ different orders, so that the recurrence relation

$$p(n) = np(n - 1)$$

is obtained. This, with the boundary condition $p(1) = 1$ gives, by Example 1.1,

$$p(n) = n!. \qquad (1.1)$$

Example 1.1:

A breakfast cereal competition lists 10 properties of a new make of car and asks the eater to place these properties in order of importance. (a) How many orderings are possible? (b) How many would be possible if the first and tenth places were already specified?

Solution:

(a) $10\,!$;

(b) 8 properties are left to be ordered. This can be done in $8\,!$ ways.

Example 1.2:

A sports magazine decides to publish articles on all 22 first division (football) league clubs, one club per week for 22 weeks. In how many ways can this be done if the first article must be about Arsenal? How many if Wolves and Stoke must be featured on consecutive weeks?

Solution:

(a) 21 teams are left to be ordered, so there are $21\,!$ orderings.

(b) Consider Stoke-Wolves as one unit. Then this unit and 20 others have to be ordered. This can be done in $21\,!$ ways. But in each way there are two possible orderings of the Stoke-Wolves unit, so the required number is $2 \times 21\,!$

Exercises 1:

1. How many 9 -digit numbers can be obtained by using each of the digits $1,2, \ldots, 9$ exactly once? How many of these are bigger than 500000000 ?
2. How many permutations are there of the 26 letters of the alphabet in which the 5 vowels are in consecutive places?
3. How many ways are there of listing the 26 letters of the alphabet (a) so that the five vowels appear consecutively, (b) so that A and B do not occur next to each other?

4. It is required to seat n people round a table. Show that this can be done in $(n-1)$! different ways. (Hint: put the n people in a row and then join up the ends of the row. Some rows will give the same circular arrangement.)
5. How many different necklaces can be designed from n colours, using one bead of each colour?

1.2. Ordered selections:

The competition of Example 1.1 will now be slightly changed. Suppose the eater is now asked to choose only the 6 most important properties and to place these 6 in order of importance. How many possible lists are there now?

In general, let $p(n,r)$ denote the number of ways of listing r objects chosen from n. As for permutations above, the first object on the list can be chosen in n ways, and then $(r-1)$ of the remaining $(n-1)$ objects have to be added to the list. Thus

$$p(n,r) = np(n-1,r-1).$$

This gives

$$p(n,r) = n(n-1)\dots(n-r+2)p(n-r+1,1)$$

where there is the boundary condition $p(s,1) = s$ for all s. Thus

$$p(n,r) = n(n-1)\dots(n-r+2)(n-r+1) = \frac{n!}{(n-r)!} \quad (2.2)$$

In the above example, therefore, the number of possible lists is $\frac{10!}{4!}$.

Example 1.3:

There are 5 seats in a row available, but 12 people to choose from. How many different seatings are possible?

Solution:

$p(12,5) = \frac{12!}{7!}$.

Example 1.4:

30 girls, including Miss U.K., enter a Miss World competition. The first 6 places are announced.

(a) How many different announcements are possible?

(b) How many if Miss U.K. is assured of a place in the first six?

Solution:

(a) $p(30,6) = \frac{30!}{24!}$;

(b) Here subtract from $p(30,6)$ the number of placings which do not include Miss U.K. Such placings are in effect ordered selections of 6 from 29 candidates, so there are $p(29,6)$ such orderings. The required number is therefore

$$\frac{30!}{24!} - \frac{29!}{23!} = \frac{29!}{24!}(30 - 24) = \frac{6.29!}{24!}.$$

In the above examples, once an object has been chosen it cannot be chosen again. However, sometimes repetitions are allowed.

Example 1.5:

For each day of the 5-day working week I can choose any one of 4 newspapers to read in the train. How many different buys are possible in a week?

Solution:

The point here is that if I buy the Times on Monday, I can still buy the Times later on in the week. For each day there are 4 choices, so the total number of choices for the week is

$$4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$$

Clearly the number of ways of choosing k objects in order from a set of n objects, with repetitions allowed, is just n^k, since there are n objects to choose from each time.

Exercises 2:

1. Evaluate $p(7,4), p(8,2), p(9,5)$.
2. A car registration number is to consist of 3 letters followed by a number between 1 and 999 . How many car numbers are possible?
3. Tom has 75 books but enough room on his bookshelf for only 20 . In how many ways can he fill his shelf?
4. How many numbers between 1000 and 3000 can be formed from the digits 1,2,3,4,5 if repetition of digits is (a) allowed, (b) not allowed?
5. In twelve-tone music, the twelve notes of the chromatic scale are put in a row, and have to be played in that particular order. How many rows are possible?
6. A 12-person committee has to appoint from its own members a chairman, secretary and treasurer. In how many ways can this be done?
7. In how many ways can a 5-letter word be formed from an alphabet of 26 letters if repetitions are (a) allowed, (b) not allowed?
8. A binary sequence of length n is a string of n digits each of which is 0 or 1 . How many such sequences are there? List all those of length 4.

1.3. Unordered selections:

Often in making a selection, the selected objects are not placed in any particular order. For example, if 5 out of 8 books are to be chosen, the only interest is in which 5 are chosen, not in the order in which they are chosen. How many ways are there of choosing 5 books from 8?

More generally, let $c(n,r)$ denote the number of ways of choosing r objects from n given objects, without taking order into account. Consider any selection of r objects. This selection can be ordered in $p(r) = r!$ different ways, and so each unordered selection gives rise to $r!$ ordered selections. Thus

$$r!\,c(n,r) = \text{ total number of ordered selections}$$
$$= \frac{n!}{(n-r)!}$$

so that

$$c(n,r) = \frac{n!}{r!\,(n-r)!} \tag{1.3}$$

This number is often written as $\binom{n}{r}$. Thus

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

For example,

$$c(8,5) = \binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 56.$$

There are therefore 56 ways of choosing 5 books from 8 .

Example 1.6:

The third method of attacking the problem gave

$$f(n,k) = c(n+k-1, n-1)$$

Thus

$$f(n,k) = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!\,(n+k-1-n+1)!} = \frac{(n+k-1)!}{(n-1)!\,k!}$$

agreeing with method 2.

One of the important properties of the numbers $\binom{n}{r}$ is given in the following theorem. The convention that $\binom{n}{0} = 1$ is followed.

Theorem 1.1:

$\binom{n}{r} = \binom{n}{n-r}$. $(0 \leqslant r \leqslant n)$.

Proof:

Two alternative proofs are given, both of which should be studied.

First proof

Selecting r objects from n is equivalent to choosing the ($n - r$) objects which shall not be selected!

Second proof

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \binom{n}{n-r}.$$

Example 1.7:

(1) $\binom{8}{3} = \binom{8}{5}$. (2) $\binom{n}{n-1} = \binom{n}{1} = n$ and $\binom{n}{n-2} = \binom{n}{2} = \frac{1}{2}n(n-1)$ for all n.

The numbers $\binom{n}{r}$ are of extreme importance in mathematics. This is because of the following theorem.

Theorem 1.2:

Let n be a positive integer. Then, if $(1+x)^n$ is expanded as a sum of powers of x, the coefficient of x^r is $\binom{n}{r}$.

Example 2.8.

$$\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
\end{aligned}$$

Proof of Theorem 1.2. Consider the product

$$(1+x)(1+x)\ldots(1+x) \ (n \text{ brackets }).$$

A term x^r is obtained by choosing r of the brackets, selecting the term x from each of them, and selecting the term 1 from the remaining ($n-r$) brackets. Thus, the number of times x^r is obtained is just the number of ways of choosing r of the n brackets, i.e. $c(n, r)$.

The coefficients in the expansions are, by Example 1.8,

$$
\begin{array}{c}
1 \\
11 \\
121 \\
1331 \\
14641 \\
15101051 \\
\vdots
\end{array}
$$

This array is known as Pascal's triangle. The ($n+1$) th row gives the numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. The property of Theorem 1.1 is simply that each row reads the same forwards as backwards. But another property is clear in the triangle: each number in the array is the sum of the two numbers immediately above it. This is because of the following recurrence relation, for which again two proofs are given.

Theorem 1.3:

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

First proof. $\binom{n}{r}$ is the number of ways of choosing r objects from n. Any particular choice may or may not include the nth object. If the nth object is included, the problem is that of choosing $(r-1)$ from the remaining $(n-1)$, and this can be done in $\binom{n-1}{r-1}$ ways. If the nth object is not chosen, r objects have to be selected from the remaining $(n-1)$, and this can be done in $\binom{n-1}{r}$ ways.

Second proof. $\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}$

$$= \frac{(n-1)!}{r!\,(n-r)!}\{r+(n-r)\} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}.$$

Example 1.9:

$\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$.

Theorem 1.2 can be re-expressed in the following form.

Theorem 1.2: If n is any positive integer, then

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

More generally, the following results holds.

Theorem 1.4. If n is any positive integer, then

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$
$$= \sum_{r=0}^{n}\binom{n}{r}a^{n-\mathrm{r}}b^r$$

Proof:

$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n \sum_{r=0}^{n}\binom{n}{r}\left(\frac{b}{a}\right)^r = \sum_{r=0}^{n}\binom{n}{r}b^r a^{n-r}$.

Theorem 1.4 is known as the binomial theorem, and the numbers $\binom{n}{r}$ are called the binomial coefficients. The name 'binomial' refers to the fact

that the theorem is concerned with the expansion of the nth power of a sum of two symbols. As an example of the theorem,

$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

In the language of Chapter 1, $(1+x)^n$ is the generating function of the binomial coefficients. One useful special case is the following.

Theorem 1.5:

If n is any positive integer, then

$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots + (-1)^n\binom{n}{n}x^n = \sum_{r=0}^{n} \binom{n}{r}(-1)^r x^r.$

This follows from the binomial theorem on choosing $a = 1, b = -x$.

Example 1.10.

Later on in this book, it will be necessary to consider the following series:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Using the binomial theorem, it is easy to prove the following important property:

$$\exp(x)\exp(y) = \exp(x+y).$$

For the left-hand side is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right),$$

so that, when the brackets are multiplied together, the terms of the form $x^r y^s$ with $r + s = n$ which are obtained are precisely

$$\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}\frac{y}{1!} + \frac{x^{n-2}}{(n-2)!}\frac{y^2}{2!} + \cdots + \frac{x}{1!}\frac{y^{n-1}}{(n-1)!} + \frac{y^n}{n!}$$

$$= \frac{1}{n!}\left\{x^n + \frac{n!}{(n-1)!\,1!}x^{n-1}y + \frac{n!}{(n-2)!\,2!}x^{n-2}y^2 + \cdots + y^n\right)$$

$$= \frac{1}{n!}(x+y)^n$$

Already in this book the need for an expansion for $(1-x)^{-n}$ has been met. Such an expansion is impossible if it is required that there should be only a finite number of terms, as happens in the expansion of $(1-x)^n$, but an infinite series representing $(1-x)^{-n}$ can be obtained. In fact, since the problem which gave rise to $(1-x)^{-n}$ has already been solved by another method (method 3), we can turn this to our advantage and use it to prove:

Theorem 1.6:

If n is any positive integer, then

$$(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots$$
$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r.$$

Proof. The theorem simply states that $f(n,r) = \binom{n+r-1}{r}$. But this has been proved by method 3 (see Example 2.6).

Example 1.11:

$(1-x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + \cdots$

Example 1.12:

Use Pascal's triangle and the fact that

$$f(n,r) = \binom{n+r-1}{r}$$

to extend the following table of values of $f(n,r)$.

r^n	1	2	3	4	5	6
1	1	2	3	4		
2	1	3	6	10		
3	1	4	10	20		
4						

In concluding this section, note that the first three entries in the following table have been explicitly presented in this chapter. What about the fourth? The number of unordered selections of k objects (with repititions allowed) from n objects is the number of ways of choosing $x_1, \ldots, x_n$ (where x_i is the number of times the i th object is chosen) such that $x_1 + \cdots + x_n = k$, i.e. is just $f(n,k) = \binom{n+k-1}{k}$.

Choose k from n	Number of ordered selections	Number of unordered selections
Repetitions not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
Repetitions allowed	n^k	$\binom{n+k-1}{k}$.

Exercises 3:

1. Expand $(1+x)^8$ and $(1-x)^8$.
2. Evaluate $\binom{11}{4}, \binom{13}{7}, \binom{15}{8}$.
3. Obtain the first few terms in the expansion of $(1-x)^{-8}$.
4. How many solutions are there of the equation $x + y + z = 10$ with x, y, z non-negative integers?
5. How many solutions are there of the equation $x + y + z = 10$ with x, y, z positive integers?
6. An eight-man committee is to be formed from a group of 10 Welshmen and 15 Englishmen. In how many ways can the committee be chosen if
   (a) the committee must contain 4 of each nationality,
   (b) there must be more Welshmen than Englishmen,
   (c) there must be at least two Welshmen?
7. A king is placed on the bottom left hand square of an 8×8 chessboard and is to move to the top right-hand corner square. If it can move only up or to the right, how many possible paths does it have to choose from?
8. By using the identity

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

and considering the coefficient of x^n on both sides, prove that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Verify this in the case $n = 5$.

Unit II

Parings problems: Pairings with in a set-Pairing between sets

Chapter 2: Sections 2.1 and 2.2

2.1. Pairings within a set

Pairings problems fall, roughly speaking, into two categories. The first type is concerned with splitting up a set with an even number of elements into pairs of elements, for example arranging 2n students in n pairs to share rooms in a college residence hall. The second type is concerned with pairing off the elements of one set with those of another, for example assigning jobs to applicants so that no two applicants get the same job. A problem of the first type will serve as the starting point of this chapter. Given 2n objects, how many ways are there of forming n pairs?

Example 1:

Six men A, B, C, D, E, F are to be paired off. One way is A with B, C with D, E with F, whereas another way is A with C, B with F, E with D. There are 15 possible ways altogether and the reader is left to produce the remaining 13. This method is rather lengthy, and so a better method is looked for.

In general, when there are 2n objects, a first idea might be to place these objects in brackets (2 in each) strung in a row as shown.

(,) (,)……(,)

The objects can be placed in the spaces in (2n)! different ways. In each bracket, however, there are 2! different orderings, which have to be considered as giving the same pairing, so the number (2n)! must be divided by 2! for each bracket, i.e. by $(2!)^n$. Further, the order of the brackets does not matter, and since the brackets can be arranged in n! different ways, each distinct pairing has in fact been obtained n! times. On dividing by n! the total number of different pairings is finally

$$\frac{(2n)!}{(2!)^n n!} \quad \ldots\ldots\ldots.. (2.1)$$

For example, n = 3 gives $\frac{6!}{2^3 3!} = 15$ as already observed. This method oa clearly generalizes to the problem of splitting up mn objects into n sets of m objects, the working above simply corresponding to m = 2.

Theorem 2.1:

Let S be a set of mn objects. Then S can be split up (partitioned) into n sets of m elements in $\frac{(mn)!}{(m!)^n n!}$ different ways.

Proof:

Replace 2 by m in the above argument.

Example 2.2:

A wholesale company has to supervise sales in 20 towns. Five members of staff are available, and each is to be assigned 4 towns to supervise.

(a) In how many ways can the 20 towns be put into 5 groups of 4?

(b) In how many ways can the towns be assigned to the staff?

Solution:

(a) The theorem gives the number as $\frac{20!}{(4!)^5 5!}$

(b) Imagine that the towns have been arranged in 5 groups of 4 in some particular way. Then the 5 groups can be assigned to the 5 men in 5! different ways, depending on which group goes to the first man, which to the second, and so on. The required number is therefore 5! times the number in part (a), i.e. $\frac{20!}{(4!)^5 5!}$.

Note on (a). This corresponds to omitting the last part of the argument which proved (2.1). Here the order of the brackets does matter, since the first bracket corresponds to the first man, and so on.

In practice, of course, things are generally far more complicated. In Example 2.1, for example, A may refuse to be paired with B. This gives a new problem. Instead of asking how many pairings are possible, the question becomes: does even one pairing exist, taking into account the likes and dislikes of the six people?

Example 2.3.

In Fig. 2.1, the 6 dots represent 6 people. Two dots are joined by a line if and only if the two people represented by the dots are willing to be paired together. Is it possible to achieve a pairing?

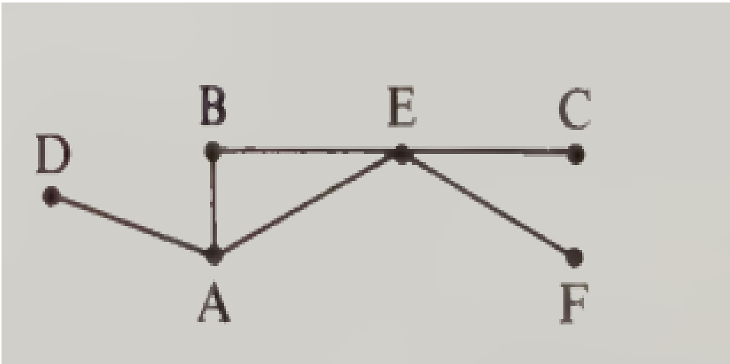


Figure 2.1.

Solution:

No. For C can be paired only with E, and this leaves no-one to be paired with F. A diagram such as Fig. 2.1 is called a graph. The dots are known as the vertices and the lines as the edges. The convention will be followed in this book that any pair of vertices of a graph can be joined by at most one edge. Roughly speaking, the failure to find a pairing in the above example is due to the lack of edges. Vertex F has only one edge emanating from it, whereas it could have as many as five. The following theorem shows that if each vertex has at least half the possible edges from it present, then a pairing can be achieved. The proof is constructive in the sense that it not only proves there is a pairing but it describes how a pairing can be found in practice by a routine procedure. Such a procedure is called an algorithm, and can be programmed for a computer. Two definitions are given before the proof. The degree of a vertex of a graph is the number of edges with that vertex as an end-point. For example, in Fig. 2.1, the respective degrees are 3, 2, 1, 1, 4, 1. Also, a pairing off of all the vertices of a graph is often called a complete or perfect matching of the graph. Clearly a graph needs to have an even number of vertices if it is to have a perfect matching.

Theorem 2.2:

If a graph has 2n vertices, each of degree ≥n, then the graph has a perfect matching.

Proof:

Assuming that r pairs of vertices have so far been paired off, where r<n, the proof shows how to increase this to (r + 1) pairs. If there are two vertices not yet paired off but joined by an edge, they can be taken immediately as the (r + 1)th pair. So suppose now that no two of the remaining vertices are joined by an edge. Choose any two of them, and call them a and b. It will now be shown that there must be a pair u, v of vertices already paired together such that a and uw are joined by an edge and b and v are joined by an edge (see Fig. 2.2). The pairings can then be rearranged so that a is paired with u and b with v, thus increasing the number of pairs to (r + 1).

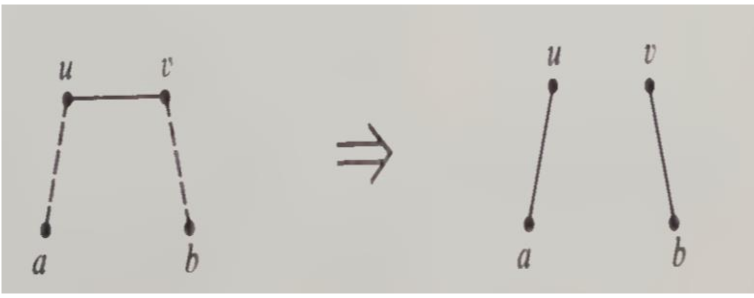


Figure 2.2.

Suppose no pair u, v exists. Then each of the r pairs x, y of vertices so far formed is such that at most two of the four possible edges ax, ay, bx, by actually appear in the graph. Thus, the total number of edges from the r pairs to a and b is at most

2r < 2n. But since a-and b are both of degree ≥n, the number must be ≥ 2n, giving a contradiction. The algorithm is therefore as follows. Having obtained r pairs, scan the remaining (2n — 2r) vertices to see if two of them are joined by an edge. If not, choose any two of them, a, b, and scan the pairs x, y already formed until one is found such that a is joined to x and b to y. Then replace the pair x, y by the two pairs a, x and b, y. If r + 1< n, repeat the whole process.

Exercises 2.1:

1. 10 people meet and form 5 pairs. In how many ways can these 5 pairs be formed?
2. 16 teams qualify for a particular round of the F.A. Cup. How many possible pairings are there for the 8 games if it (a) is(b) is not taken into account which teams are drawn at home?
3. A pack of 52 cards is divided among 4 people so that each gets 13 cards (as in bridge). How many such deals are possible?
4. In the Scottish League Cup, 16 first division clubs were arranged in 4 groups of 4. In how many ways can this be done? Recently, Rangers and Celtic were drawn in the same section two years running. Show that this is not as strange as the press made it out to be by finding the number of ways the draw can be made with Rangers and Celtic in the same section, and verifying that this number is precisely one fifth of the total number of possible draws.
5. The following are all the allowable pairings of 8 objects. (1, 2), (1, 3), (2, 4), (2, 5), (4, 6), (5, 7), (5, 8), (7, 8). Obtain a complete matching.
6. Draw a graph with 10 vertices, each of degree ≥ 5, and find a perfect matching for it.
7. Construct a graph with 10 vertices, each of degree ≥ 4, with no perfect matching.

2.2. Pairings between sets

A number of jobs are available in a large industrial organization, and applicants are examined for suitability for each of the jobs. In what circumstances is it possible to assign a suitable person to each?

This problem, one type of assignment problem, is typical of those to be examined in this section. More generally, given two sets A, B (here, the set of jobs and the set of applicants), when is it possible to pair off each member of A with a different member of B?

Example 2.4:

Five jobs are available. For each z= 1, …..,5, let S_i denote the set of applicants suited for the ith job. Can all the jobs be filled?

$S_1 = \{$ (A, B, C}, $S_2 =$ {D, E} S_3 ={D} S_4 = {E}, S_5 = {A, E}.

Solution:

No. The second, third, and fourth jobs have only 2 suitable applicants between them. But 2 men cannot fill 3 jobs.

This example deserves closer scrutiny. By introducing the sets S_1,,..., S_5, the problem has been re-expressed as one of the following type.

Given sets S_1,..., S_n ,is it possible to choose a different element from each set S_i?

If it is possible, then the chosen elements are called distinct representatives of the sets. In the above example, the sets S_3, S_4, S_5 possess distinct representatives (D, E, and A, in that order), but the sets S_1, S_2, S_3, S_4, S_5 do not.

The reason is:

There are 3 sets containing between them less than 3 elements. Clearly, if distinct representatives do exist, then, for every value of k;

any k sets contain between them at least k elements. ……… (2.2)

This is a necessary condition. The interesting and useful fact is that the condition is not only necessary but it is also sufficient. In other words, if (2.2) holds for every value of k, then it is guaranteed that distinct representatives can be found. The proof which will be given is an algorithm which not only shows that distinct representatives exist, but gives a method of actually finding them. The following result will then have been proved.

Example 2.5:

If $A_1 = \{1,2\}$; $A_2 = \{4\}$; $A_3 = \{1,3\}$; $A_4 = \{2,3,4\}$. Find the distinct representative for the set A_i.

Sets	Distinct representative	
A_1	1	2
A_2	4	4
A_3	3	1
A_4	2	3

Example 2.6:

Find the set for distinct representative for the set {a};{a,b,c};{c,d,e}; {b, d,e};{a,d,g};{f};{c,f}

Sets	Distinct representative	
$\{a\}$	a	a
$\{a,b,c\}$	b	b
$\{c,d,e\}$	e	d
$\{b,d,e\}$	d	e
$\{a,d,g\}$	g	g
$\{f\}$	f	f
$\{c,f\}$	c	c

Theorem 2.3:

(Philip Hall's theorem on distinct representatives). The sets A_1,,..., A_5, possess a system of distinct representatives if and only if, for all k =1,...,n, any k $A_i s$ contain at least k elements in their union.

An alternative formulation would be:

Assignment Theorem:

This assignment problem has a solution if and only if there is no value of k for which there are k jobs with fewer than k suitable applicants between them.

Replacing the job situation by marriage gives yet another formulation of Hall's theorem which has earned it the popular title of the Marriage Problem.

Marriage Theorem:

Given a set of men and a set of women, each man makes a list of the women he is willing to marry. Then each man can be married off to a woman on his list if and only if,

(*) {for every value of k, any k lists contain in their union at least k names.

Proof:

It is shown how, on the assumption that r <n men have been paired off with suitable ladies, to increase this to (r + 1) men.

Suppose r men have been paired off. If there is a man left who has on his list a woman who is still unattached, an (r + 1)th pairing is immediate.

So, suppose that all women on remaining lists are already attached.

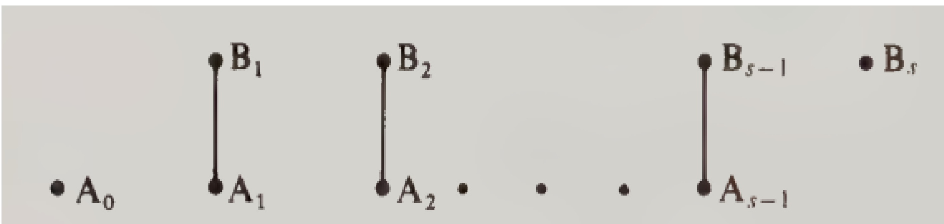


Figure 2.3.

Choose any unmarried man A_0 (see Fig. 2.3). By (*) with k = 1, there is a woman B_1 on his list. B_1 is married to A_1, say by (*) with k = 2, the combined lists of A_0 and A_1 contain the name of at least one more woman B_2. If B_2 is unmarried, stop. If B_2 is married to A_2, then, by (*) with k = 3, the combined lists of A_0, A_1, A_2 contain a third name, say B_3 . If B_3 is unmarried, stop. If B_3 is married to A3, repeat the process, and continue until an unmarried woman B_s is reached. (This must happen eventually since not all the women are married, and no B_i occurs twice in the process.

Note that, by construction, each B_i is on the list of at least one A_j with j< i. This is very important. Consider now B_s Pair her off with an A_i on whose list she appears (i <s). This frees B_j.Next pair off B_i with an A_j (j <i) on whose list she appears. This frees B_j. Repeat until some B is freed and re-paired with A_0. This must eventually happen. Then take all the new pairings and all the original ones which have not been tampered with. Now (r+ 1) pairs have been obtained. Repeat the process if r + 1<n.

This constructive proof, which was communicated to the author by D. J. Shoesmith, has the advantage that the conditions (*) need not be checked before the construction is attempted. If (*) does not hold,

this will become clear when the method breaks down. On the other hand, if (*) holds, the method will not break down.

Application to Latin squares

1	2
2	1

1	2	3
2	3	1
3	1	2

1	2	3	4
2	1	4	3
3	4	1	2
4	3	1	2

The three squares each possess the following properties (for n = 2, 3, 4 respectively),

(1) each row contains each of the numbers 1, .. ., 1 exactly once;

(2) each column contains each of 1, ..., n exactly once.

These are the properties which characterize Latin squares. An n x n Latin square based on the numbers 1, . . ., 1 is thus defined to be an array of n rows and n columns satisfying properties (1) and (2) above. Apart from their intrinsic mathematical charm, Latin squares do have their uses, and may first have been studied because of their uses in the design of experiments. A simple introduction to this topic can be found in Fisher's classical book [1].

How are Latin squares constructed? Very easily, in fact, for the next result shows that the construction can be carried out a row at a time. The proof requires the idea of a Latin rectangle, which is simply a rectangular array with r rows and n columns ($r \leq n$) in which

(1) each row contains each of 1, ...,n exactly once;

(3) no column contains a number more than once.

For example, the first three rows of the 4 x 4 Latin square above give a 3 x 4 Latin rectangle. The theorem to be proved is essentially a converse of this result: it states that any Latin rectangle can be made into a Latin square by adding further rows, without having to alter the rows already there.

Theorem 2.4:

If r <n, any rx n Latin rectangle can be extended to an (r + 1) xn Latin rectangle.

Proof:

An (r + 1)th row has to be added, the jth number in which does not yet occur in the jth column of the rectangle. This suggests that for each j=1,...,n the set S; should be defined as follows:

S_j = set of numbers between | and n which have not yet appeared in the jth column.

To prove the theorem, it is sufficient to show that the sets S; possess distinct representatives. These distinct representatives will form the next row.

Suppose then that the sets S; do not possess distinct representatives. Then by Hall's theorem, there must, for some k, be k sets S; which in their union contain less than k numbers. Now clearly each set S; has (n — r) elements, so these k sets contain between them k(n — r) numbers, not taking repetitions into account. How many times can a number be repeated? Each number has occurred exactly once in each row, and hence in exactly r of the columns. Each number therefore occurs in exactly (n —r) of the sets S_i The k sets therefore contain k(n — r) elements, with no element repeated more than (n — r) times, and so must contain at least k distinct elements. This gives a contradiction, and the proof is complete.

Application to tournaments

Consider a tournament involving n teams in which every team plays against every other team exactly once. Such a tournament will have $\binom{n}{2}$ f games and is often called a round-robin tournament. Suppose that each game produces a winner which is awarded one point and a loser which gains no point. Then after all $\binom{n}{2}$ games have been played the final points obtained by each team, when written in decreasing order, form what is called the score sequence of the tournament.

Example 2.5.

A beats B, A beats C, B beats C, D beats A, B beats D, D beats C. So, A, B, and D finish with 2 points each and C finishes with none. The score sequence is therefore (2, 2, 2, 0).

Which sequences of m non-negative integers can be realized as the score sequence of some tournament? For example, is it possible to have a tournament with 6 teams, with score sequence (5, 4, 4, 1, 1, 0)? Certainly, the sum of the scores is 15 =$\binom{6}{2}$, However, to see that the answer is no, note that the three bottom teams have only two points between them, whereas they should have at least three points between them since they play 3 =$\binom{3}{2}$, games amongst themselves. In general, a sequence $(a_1, a_2, \dots \dots . a_n), a_1 \geq a_2 \geq \cdots \dots \geq a_n$, of non-negative integers can be a score sequence only if

$$a_1 + a_2 + \cdots \dots . + a_n = \binom{n}{2} \qquad \dots\dots\dots\dots (2.3)$$

And $a_{n-r+1} + a_{n-r+2} + \cdots \dots . + a_n \geq \binom{r}{2}$ for each r, $2 \leq r \leq n$. …… (2.4)

The remarkable fact is that these obviously necessary conditions are also sufficient.

Theorem 2.5. (Landau's theorem).

The non-negative integers

as form the score sequence of a tournament if and only if conditions (2.3) and (2.4) are satisfied.

This result will be deduced from Hall's marriage theorem. In fact, a slight generalization of Hall's theorem is needed, which can be expressed in terms of harems rather than marriages; here each man can marry more than one woman, but no woman can have more than one husband.

Theorem 2.6. (The Harem theorem)

Let w_1,,..., w_n, be non-negative integers, and suppose that men M_1,,..., M_n, each makes a list of the women he is willing to marry. Then each M_i, can be married to w_i, women on his list if and only if, for any subset $\{i_1$,,..., $i_r\}$of {1,...,n}, the lists of men $M_{i_1}, \dots \dots . M_{i_r}$ contain in their union at least $w_{i_1}, \dots \dots . w_{i_r}$names.

Proof:

The condition is clearly necessary, so we prove sufficiency. Replace each man M_i by w_i copies, each of which has the same list as M_i had. The problem is to pair off each copy with a woman on his list. Consider any set of copies, consisting of, say, x_i copies of $M_i$where $x_i \leq w_i$, $i \in I \subseteq \{1, \dots \dots n\}$.Their Their lists contain at least $\sum_{i \in I} w_i \geq \sum_{i \in I} x_i$names, names as men, so by Hall's theorem the copies can be married off.

Exercises:

1. If A, = {1, 2}, Az = {4}, A3 = {I, 3}, and A, = {2, 3, 4}, find distinct representatives for the sets A_i.
2. Find a set of distinct representatives for the following sets: {a}, {a, b, c}, {c, d}, {b, d, e}, {e, f}, {a, d, g}, {f}.
3. Construct 2 different 5 x 5 Latin squares which have the same first rows, but no other rows the same.
4. mn newspaper reporters each cover one sport and one foreign country, in such a way that each of n sports has m reporters and each of n countries has m reporters. Use the previous example to show that it is possible to staff m newspapers each with n reporters so that each sport and each country is covered by each newspaper.

5. Does there exist a tournament with score sequence (a) (4, 4, 1, 1, 0), (b) (3, 3, 3, 1, 0)? If yes, construct one: if no, explain why.

Unit III

Recurrence-Fibonacci-type relations using generating functions- Miscellaneous methods.

Chapter 3: Sections 3.1- 3.4

3.Recurrence

3.1. Some miscellaneous problems:

Some combinatorial problems reduce to examining a sequence $\{a_n\}$ of numbers $a_1, a_2, a_3, \ldots$ in the hope of obtaining a formula for the nth member a_n of the sequence. Often a_n is expressed in terms of previous members of the sequence, i.e. a recurrence relation is given, and also the first few values are given, for example a_1 and a_2. The problem is then to deduce a formula for a_n.

A few such problems are now exhibited.

Example 3.1:

The Fibonacci sequence, mentioned in Chapter 1, is defined by

$$a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2} \ (n \geqslant 3),$$

and the problem is to find a formula for a_n. This sequence was investigated in the 13th century by Leonardo Fibonacci of Pisa, in connection with the growth of the rabbit population.

Example 3.2:

Some combinatorial problems in chemistry reduce to counting the number of graphs of a certain type. A tree is defined to be a connected graph with no cycles, i.e. a connected graph in which it is impossible to start at a vertex, move along different edges and arrive back at the starting place. Examples of trees are shown in Fig. 3.1,

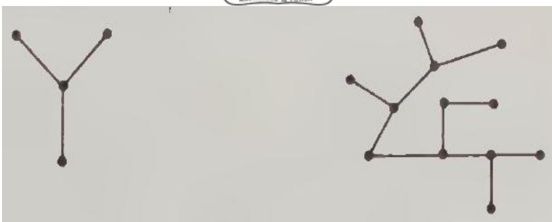


Fig. 3.1

whereas the graph in Fig. 3.2 is not a tree, one cycle being $abcd$. Trees can be used to represent the structure of chemical compounds, and it was in this way that Cayley was led to his studies of graph theory in the 1870 s.

As an example of the type of problem involved, consider the problem of counting simple rooted trees. A simple tree is defined to be a tree in

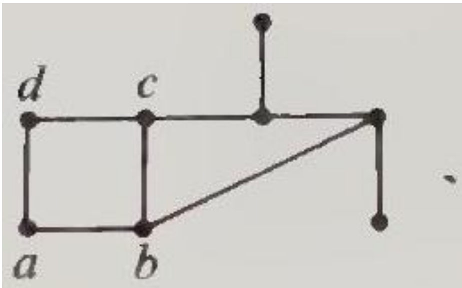


Fig. 3.2

which each vertex is of degree $\leqslant 3$. (Recall that the degree of a vertex is the number of edges emanating from it.) One way of looking at a simple tree is to consider it as a road system in which one has a choice of at most two roads at each roadend. The simple trees to be considered are those rooted at a certain vertex P (see Fig. 3.3). P can be considered as the

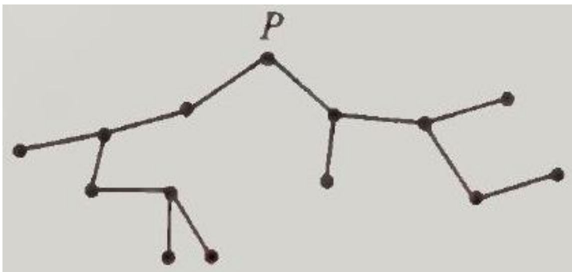


Fig. 3.3

starting point of the tree's growth, and in accordance with the requirement that at most two edges are available on reaching any vertex, it will be assumed that there are at most two edges emanating from P. An example of such a rooted simple tree is shown. The problem is to evaluate u_n, the number of different rooted simple trees with n vertices.

A difficulty, fundamental to most combinatorial problems, immediately arises. When are two trees to be considered different? For example, are the two trees in Fig. 3.4 the same or different? After all, in any practical realization, (a) can be picked up and turned over to give (b).

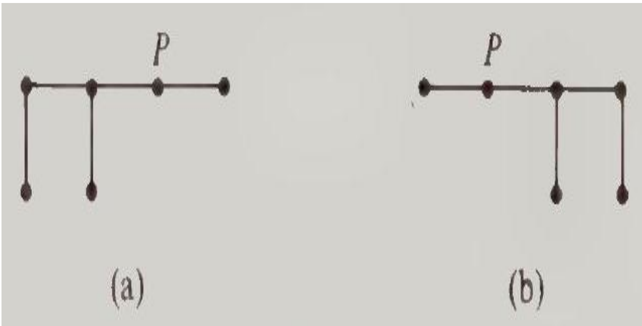
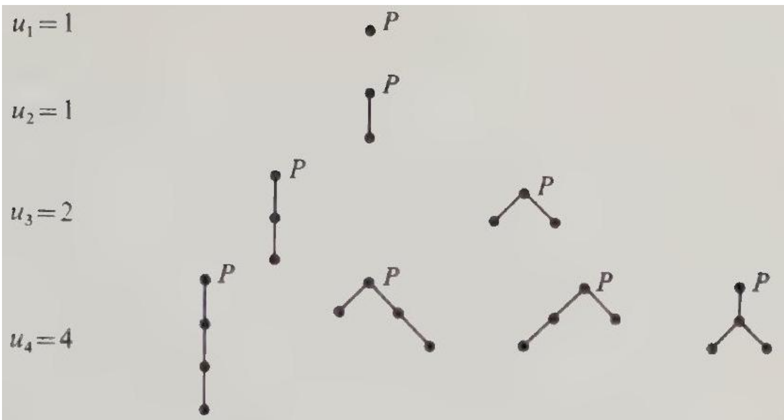


Fig. 3.4

Since it is a good idea to start with as simple a problem as possible, it will be considered here that (a) and (b) are distinct. Then

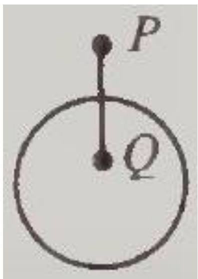


and so on. In evaluating $u_n, n \geqslant 2$, two types of trees have to be considered. Let

s_n denote the number with only one edge from the root P, and let d_n denote the number with two edges from P. Clearly

$$u_n = s_n + d_n. (n \geqslant 2) \qquad (3.1)$$

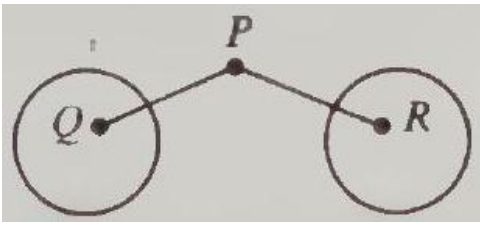
Now consider s_{n+1}. A tree contributing to s_{n+1} is of the form



where, inside the circle, there can be any simple tree with n vertices rooted at Q. There are u_n such trees. Thus

$$s_{n+1} = u_n. \qquad (3.2)$$

Next consider d_{n+1}. A tree contributing to d_{n+1} is of the form



where there is a rooted simple tree at Q with, say, r vertices, and a rooted simple tree at R with s vertices, $r + s = n, r \geqslant 1, s \geqslant 1$. For each such pair of values of r and s there are u_r choices of what happens at Q and u_s choices at R; $u_r u_s$ choices altogether. Thus

$$d_{n+1} = u_1u_{n-1} + u_2u_{n-2} + \cdots + u_{n-1}u_1$$
$$= \sum_{\substack{r+s=n \\ r\geqslant 1, s\geqslant 1}} u_ru_s. \qquad \ldots\ldots\ldots\ldots \qquad (3.3)$$

The relations (4.1) to (4.3) then yield

$$u_{n+1} = s_{n+1} + d_{n+1}$$
$$= u_n + \sum_{\substack{r+s=n \\ r\geqslant 1, s\geqslant 1}} u_ru_s,$$

i.e.

$$u_n = u_{n-1} + (u_1u_{n-2} + u_2u_{n-3} + \cdots + u_{n-2}u_1). \qquad (3.4)$$

How is a formula for u_n obtained from this recurrence relation?

Example 3.3:

The problem of derangements. Suppose that n jobs have been assigned to n people. In how many ways can they be reassigned the following day so that no person is given the same job as before?

In general, a derangement of the numbers $1,2,\ldots,n$ is a rearrangement or permutation of them such that no number appears in its original position. For example, 23514 is a derangement of 12345, but 23541 is not. Let a_n denote the required number; then a_n is simply the number of derangements of $1,\ldots,n$, for it can be supposed that the jobs are so labelled that the i th person got the i th job on the first day.

Clearly $a_1 = 0$ (why?), $a_2 = 1, a_3 = 2$. To see that $a_3 = 2$, note that the only derangements of 123 are 231 and 312.

Suppose now that $n > 2$, and consider two possibilities. The first possibility is that in a derangement of $1,\ldots,n$ the number n changes places with some other number r. There are $(\ n-1\ )$ choices for r, and for each such choice the remaining $(\ n-2\ )$ numbers must undergo a derangement. The number of ways this can happen is $(n-1)a_{n-2}$, and this therefore gives the number of derangements of $1, \dots, n$ in which n changes places with another number. The second possibility to consider is when some number r moves to the nth place, but n does not move to the r th place. In this case, ignore r which has now been placed, and relabel n by r. This gives $(n-1)$ numbers $1, \dots, (n-1)$ to arrange, and the condition is again simply that no i is to placed in the i th place. There are a_{n-1} such derangements for each of the $n-1$ choices of r, and so $(n-1)a_{n-1}$ derangements of this type. Thus

$$a_n = (n-1)a_{n-1} + (n-1)a_{n-2} \tag{3.5}$$

and the problem is how to solve this recurrence relation subject to the boundary conditions $a_1 = 0, a_2 = 1$.

Exercises 1:

1. Use (4.4) to find u_5, and check your answer by drawing all possible rooted simple trees with 5 vertices.
2. Use (4.5) to find a_4, and check your answer by writing down all the possible derangements of 1234.
3. Suppose that any newborn pair of rabbits will produce their first pair of offspring after two months, and thereafter will produce one pair per month. Starting with one newborn pair, the growth of population is as follows, where A denotes a newborn pair, B a month-old pair, and C a fully-adult pair:

after 1 month	B
2 months	CA
3	CBA
4	$CCBAA$

Prove that a_n, the number of pairs of rabbits in the population after n months, satisfies $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2} (n \geqslant 3)$. It is to be assumed that no deaths occur!

3.2. Fibonacci-type relations:

A method of solving recurrence relations of the form

$$a_n = Aa_{n-1} + Ba_{n-2}\ (n \geqslant 3) \tag{3.6}$$

is now given, where A and B are non-zero constants. As is shown, the method is essentially just that of solving the associated quadratic equation

$$x^2 = Ax + B.$$

Theorem 3.1.

Suppose that a_1 and a_2 are given and that (3.6) holds. Then

(1) if the roots α, β of the equation $x^2 = Ax + B$ are distinct, then

$$a_n = K_1\alpha^n + K_2\beta^n$$

where the constants K_1, K_2 are determined uniquely by a_1 and a_2;

(2) if $x^2 = Ax + B$ has repeated root α, then

$$a_n = (K_1 + nK_2)\alpha^n$$

Example 3.4:

The Fibonacci sequence. Here $A = B = 1$, so consider the equation $x^2 = x + 1$. This has roots

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \beta = \frac{1}{2}(1 - \sqrt{5})$$

so that

$$a_n = K_1\left(\frac{1 + \sqrt{5}}{2}\right)^n + K_2\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

for some constants K_1, K_2. Since $a_1 = 1$ and $a_2 = 2$,

$$1 = K_1\left(\frac{1+\sqrt{5}}{2}\right) + K_2\left(\frac{1-\sqrt{5}}{2}\right)$$

and

$$2 = K_1\left(\frac{3+\sqrt{5}}{2}\right) + K_2\left(\frac{3-\sqrt{5}}{2}\right)$$

These give

$$K_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}, K_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

so that

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

This may at first sight seem rather odd, since it is known that a_n must be an integer. However, all the $\sqrt{5}$ terms cancel out. The binomial theorem gives

$$\begin{aligned} a_n &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{n+1}}\left\{\sum_{r=0}^{n+1}\binom{n+1}{r}5^{r/2} - \sum_{r=0}^{n+1}\binom{n+1}{r}(-1)^r 5^{r/2}\right\} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^n}\left\{\binom{n+1}{1}5^{1/2} + \binom{n+1}{3}5^{3/2} + \binom{n+1}{5}5^{5/2} + \cdots\right\} \\ &= \frac{1}{2^n}\left(\binom{n+1}{1} + 5\binom{n+1}{3} + 5^2\binom{n+1}{5} + \cdots\right\} \end{aligned}$$

which is an interesting result since it is not immediately obvious that the sum in the brackets must be divisible by 2^n.

Note, further, that since $0 < \frac{1}{2}(\sqrt{5}-1) < 1$, $\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \to 0$ as $n \to \infty$; so a_n is approximated by $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$; in fact a_n is just the integer nearest to this value.

Proof of Theorem 3.1. (1) The idea is to explore when $a_n = a^n$ can be a solution of (3.6). Now $a_n = \alpha^n$ satisfies (4.6) precisely when $\alpha^n = A\alpha^{n-1} + B\alpha^{n-2}$, i.e. when $\alpha^2 = A\alpha + B$, i.e. when α is a root of the quadratic equation $x^2 = Ax + B$. Thus, if the quadratic equation has two distinct roots α, β, then $a_n = \alpha^n$ and $a_n = \beta^n$ are both solutions of (3.6). It follows that if K_1 and K_2 are constants then $a_n = K_1\alpha^n + K_2\beta^n$ is also a solution; for

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} &= A(K_1\alpha^{n-1} + K_2\beta^{n-1}) + B(K_1\alpha^{n-2} + K_2\beta^{n-2}) \\ &= K_1(A\alpha^{n-1} + B\alpha^{n-2}) + K_2(A\beta^{n-1} + B\beta^{n-2}) \\ &= K_1\alpha^n + K_2\beta^n = a_n \end{aligned}$$

The values of a_1 and a_2 will determine K_1 and K_2 uniquely; for the equations $a_1 = K_1\alpha + K_2\beta, a_2 = K_1\alpha^2 + K_2\beta^2$ have solution

$$K_1 = \frac{a_1\beta - a_2}{\alpha(\beta - \alpha)}, K_2 = \frac{a_1\alpha - a_2}{\beta(\alpha - \beta)}$$

(Note that $\alpha - \beta \neq 0$ since $\alpha \neq \beta$, and $\alpha, \beta \neq 0$ since $B \neq 0$.)

(2) In this case it is sufficient to verify that, if α is a repeated root of the quadratic, then $a_n = n\alpha^n$ also satisfies (4.6). Note that

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} &= A(n-1)\alpha^{n-1} + B(n-2)\alpha^{n-2} \\ &= n(A\alpha^{n-1} + B\alpha^{n-2}) - \alpha^{n-2}(\alpha A + 2B) \\ &= n\alpha^n - A\alpha^{n-1} - 2B\alpha^{n-2} \end{aligned}$$

But if α is a repeated root of $x^2 = Ax + B$ then $x^2 - Ax - B = (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2$, so that $A = 2\alpha$ and $B = -\alpha^2$; so

$$Aa_{n-1} + Ba_{n-2} = n\alpha^n - 2\alpha^n + 2\alpha^n = n\alpha^n = a_n$$

as required.

Note that this theorem and its proof will easily generalize to recurrence relations of the form

$$a_n = A_1a_{n-1} + A_2a_{n-2} + \cdots + A_ra_{n-r}$$

for any fixed r : for example, if $x^3 = Ax^2 + Bx + C$ has 3 distinct roots α, β, γ then $a_n = K_1\alpha^n + K_2\beta^n + K_3\gamma^n$ will be a solution of

$$a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$$

Example 3.5:

Solve $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ given that $a_1 = 2$, $a_2 = 6, a_3 = 20$.

Solution. The equation $x^3 = 6x^2 - 11x + 6$ has solutions $x = 1,2,3$; so the general solution of the recurrence relation is

$$a_n = A1^n + B2^n + C3^n$$

The given boundary conditions give $2 = A + 2B + 3C, 6 = A + 4B + 9C$, $20 = A + 8B + 27C$, so that $A = C = 1, B = -1$. Thus, the solution is

$$a_n = 1 - 2^n + 3^n$$

Exercises 2:

1. If $a_n = 4(a_{n-1} - a_{n-2})$ for each $n \geqslant 3$, and if $a_1 = 0, a_2 = 4$, find a_n.
2. If $a_n = 5a_{n-1} - 6a_{n-2}$ for each $n \geqslant 3$, and if $a_1 = a_2 = 1$, find a_n.
3. If a_n denotes the nth Fibonacci number, prove that
$$a_{n+2} = a_n + a_{n-1} + \cdots + a_1 + 2$$
4. Let $b_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$ Verify that $b_1 = 1, b_2 = 2$, and show that $b_n = b_{n-1} + b_{n-2} (n \geqslant 3)$. Thus b_n gives another formula for the Fibonacci numbers.
5. If a_n is the nth Fibonacci number, prove that
$$a_n^2 - a_{n-1}a_{n+1} = (-1)^n$$
6. In working through a problem, a man is said to be at the nth stage if he is n steps from the solution. At any stage he has 5 choices. Two of these result in him going to the ($n - 1$)th stage, and three of them are better in that they take him direct to the ($n - 2$) th stage. Let a_n denote the number of ways he can reach the solution from the nth stage. If $a_1 = 2$, verify that $a_2 = 7$ and obtain a recurrence relation for a_n. Deduce that

$$a_n = \frac{1}{4}\{3^{n+1} + (-1)^n\}$$

7. Find a_n if $a_n = 4a_{n-1} + 4a_{n-2} - 16a_{n-3}, a_1 = 8, a_2 = 4, a_3 = 24$.
8. Find a_n if $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, a_1 = 4, a_2 = 8, a_3 = 20$.
9. The $n \times n$ determinant D_n is defined for $n \geqslant 1$ by

$$D_n = \begin{vmatrix} 1+a^2 & a & 0 & 0 & \dots & 0 \\ a & 1+a^2 & a & 0 & \dots & 0 \\ 0 & a & 1+a^2 & a & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1+a^2 \end{vmatrix}$$

Show that, if $n \geqslant 3, D_n = (1+a^2)D_{n-1} - a^2 D_{n-2}$ and hence show that

$D_n = \frac{1-a^{2n+2}}{1-a^2}$ if $a^2 \neq 1$What if $a^2 = 1$?

10. Let a_n denote the number of n-digit sequences in which each digit is 0 or 1 , no two consecutive 0 s being allowed. Show that $a_1 = 2, a_2 = 3$ and that $a_n = a_{n-1} + a_{n-2} (n \geqslant 3)$. Hence find a_n.

11. Let b_n denote the number of n-digit sequences in which each digit is 0,1, or -1, if no two consecutive 1 s or consecutive -1 s are allowed. Prove that $b_n = 2b_{n-1} + b_{n-2} (n \geqslant 3)$ and hence find b_n.

12. A flag is to be designed with n horizontal strips each of which can be any one of the colours red, blue, green and yellow. Find the number of different designs possible in each of the following situations:

(a) there is no restriction on the colour of each stripe;

(b) no two adjacent stripes have the same colour;

(c) no two adjacent stripes have the same colour, nor do the top and bottom stripes.

13. Let a_n denote the number of ways of filling a $2 \times n$ array with 1×2 dominoes. Verify that $a_1 = 1, a_2 = 2$, and show that a_n is the nth Fibonacci number.

14. Now let a_n denote the number of ways of filling a $3 \times 2n$ array with 1×2 dominoes. Arrange the sides of length 3 to be vertical, and let x_n, y_n be, respectively, the number of solutions in which the right-hand end has, does not have, 3 dominoes touching it. Then $a_n = x_n + y_n$. Show that $x_{n+1} = x_n + y_n$ and $y_{n+1} = 2x_n + 3y_n$. Deduce that $a_{n+2} = 4a_{n+1} - a_n$ and hence obtain a formula for a_n.

15. A primitive organism takes one hour to mature. At the end of the next hour it produces two offspring and does the same each subsequent hour. Each offspring behaves in a similar fashion. Start with one newly born organism and let a_n denote the number of organisms existing after n hours. Prove that $a_n = a_{n-1} + 2a_{n-2}$ and hence find a formula for a_n.

16. Repeat the previous problem with the difference that each organism dies immediately after producing its first pair of offspring.

3.3. Using generating functions:

The recurrence relation (3.4) obtained on counting rooted simple trees does not look too attractive; it looks too difficult to deal with. It sometimes happens that such relations are better dealt with by means of generating functions. As was explained in the opening chapter, the generating function for a given sequence $a_0, a_1, a_2, \dots, a_n, \dots$ is defined to be

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where the coefficient of x^n in $f(x)$ is precisely the term a_n of the sequente.

Let $u(x), s(x)$, and $d(x)$ be the generating functions for Example 3.2, where

$$\begin{aligned} u(x) &= u_1 x + u_2 x^2 + u_3 x^3 + \cdots \\ &= x + x^2 + 2x^3 + \cdots \\ s(x) &= s_1 x + s_2 x^2 + s_3 x^3 + \cdots \\ &= x^2 + x^3 + 2x^4 + \cdots \\ d(x) &= d_1 x + d_2 x^2 + d_3 x^3 + \cdots \\ &= x^3 + 2x^4 + \cdots \end{aligned}$$

on noting that $s_1 = d_1 = d_2 = 0$ (why?). From (4.1),

$$u(x) = x + s(x) + d(x) \tag{3.7}$$

Also, since $s_{n+1} = u_n$,

$$s(x) = xu(x), \tag{3.8}$$

as can be checked by comparing the coefficients of any power of x on each side of the equation. Finally, from (3.3), it follows that

$$d(x) = x\{u(x)\}^2. \tag{3.9}$$

If (a) and (b) are considered the same, the resulting counting problem is much more difficult, and the ideas of Polya's theorem are required. Equations (3.7)-(3.9) together give

$$u(x) = x + xu(x) + x\{u(x)\}^2,$$

i.e.

$$x\{u(x)\}^2 + (x-1)u(x) + x = 0, \tag{3.10}$$

which is a quadratic equation for $u(x)$. The usual formula for solving such equations then gives

$$\begin{aligned} u(x) &= \frac{1}{2x}\left[1 - x \pm \sqrt{\{(x-1)^2 - 4x^2\}}\right] \\ &= \frac{1}{2x}\left[1 - x \pm \sqrt{\{1 - (2x + 3x^2)\}}\right] \quad (3.11) \end{aligned}$$

Now, by the binomial theorem,

$$(1-y)^{\frac{1}{2}} = 1 - \frac{1}{2}y - \frac{\frac{1}{2}\cdot\frac{1}{2}y^2}{2!} - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}y^3}{3!} - \cdots\cdots - \frac{1\cdot 3\cdot 5\ldots(2n-3)}{2^n n!}y^n - \cdots \tag{3.12}$$

so that, on taking the minus sign in (4.11) to obtain positive coefficients u_n,

$$\begin{aligned} u(x) &= \frac{1}{2x}\left[-x + \frac{1}{2}(2x + 3x^2) + \frac{1}{2^2 2!}(2x + 3x^2)^2 + \cdots\right] \\ &= x + x^2 + 2x^3 + 4x^4 + \cdots \quad (3.13) \end{aligned}$$

Technically, the problem is now solved. To find u_n, all that need be done is to read off the coefficient of x^n in (4.13). If h_n denotes the numerical value of the coefficient of y^n in (4.12), so that

$$h_n = \frac{(2n-2)!}{2^{2n-1} n!\,(n-1)!}$$

then it is straightforward to verify that

$$u_{n-1} = \frac{1}{2}\left\{h_n 2^n + h_{n-1}2^{n-2}\cdot 3\cdot\binom{n-1}{1} + h_{n-2}2^{n-4}3^2\binom{n-2}{2} + \right.$$
$$\left. + \cdots + h_{n-r}2^{n-2r}3^r\binom{n-r}{r} + \cdots\right\} \quad (3.14)$$

This formula for u_n has its unattractive side. It is not very compact, and a certain amount of effort is still required to evaluate u_n for any specific value of n, particularly when n is large. However, all that is involved is essentially the substitution of the particular value of n into (3.14), and this is all that is required of a formula. Some mathematicians would take the view that the problem of finding u_n was in fact solved well before this final formula was obtained-at the stage (3.13) of obtaining the generating function $u(x)$, since the values of all the coefficients u_n are implicit in $u(x)$. This book will take the view that an explicit formula is to be preferred to simply a generating function solution, and such a formula should be aimed at whenever possible.

Further examples on generating functions

Example 3.6. Suppose that, in the problem posed at the beginning of Chapter 1, there are 4 colours available (i.e. $n = 4$). How many colourings of the k golf balls are possible if there must be an odd number of objects coloured with the first colour?

Solution (1). As in the second approach to the original problem, the required number is the coefficient of x^k in

$$(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3 + \cdots)^3$$
$$= x(1 + x^2 + x^4 + \cdots)(1 - x)^{-3},$$

which is just the coefficient of x^{k-1} in

$$(1 + x^2 + x^4 + \cdots)\left(1 + \binom{3}{1}x + \binom{4}{2}x^2 + \binom{5}{3}x^3 + \cdots\right).$$

This coefficient is

$\binom{k+1}{k-1} + \binom{k-1}{k-3} + \binom{k-3}{k-5} + \cdots = \binom{k+1}{2} + \binom{k-1}{2} + \binom{k-3}{2} + \cdots.$

For example, $k = 6$ gives 34 possible colourings.

Solution (2). Alternatively, if exactly one ball is coloured with the first colour, there are ($k - 1$) balls left to be coloured with 3 colours. By the result (1.6), this can be done in $\binom{k-1+3-1}{k-1} = \binom{k+1}{k-1}$ ways. Similarly, if exactly 3 are coloured with the first colour, the remaining $(k - 3)$ can be coloured in $\binom{k-3+3-1}{k-3} = \binom{k-1}{k-3}$ ways. Continuing in this way the same result is obtained as before.

Example 3.7:

n-digit integer sequences are to be formed using only the integers 0,1,2,3. For example, 0031 and 3202 are two 4 -digit sequences.

(a) How many n-digit sequences are there?

(b) How many n-digit sequences have an odd number of 0 s?

Solution:

(a) The number of sequences is 4^n, since there are 4 choices for each of the n digits.

(b) This is not so easy. The difference between the problem posed here and Example 3.5 is that here it matters not only what digits appear, but also in what order they occur.

Any n-digit sequence will consist of $d_0 0$ s, $d_1 1$ s, $d_2 2$ s and $d_3 3$ s, where d_0 is odd and $d_0 + d_1 + d_2 + d_3 = n$. Any given set of d s satisfying these conditions will give rise to as many different sequences as there are ways of arranging the n numbers in a line. If the n numbers were all distinct, there would be $n!$ permutations. Thus, if the n digits are labelled so that digits of the same kind are distinguishable from one another, there are $n!$ permutations. However, two of these permutations will be the same when the labels are removed if and only if they differ only in the arrangement of the $d_0 0$ s among themselves, the $d_1 1$ s, the $d_2 2$ s, and the $d_3 3$ s.Thus each permutation of the unlabelled digits corresponds to $d_0!\, d_1!\, d_2!\, d_3!$ permutations of the labelled digits.Thus the number of distinct sequences with $d_0 0$ s, $d_1 1$ s, $d_2 2$ s, and $d_3 3$ s is

$$\frac{n!}{d_0!\, d_1!\, d_2!\, d_3!}$$

Hence the total number of sequences is equal to

$$\sum \frac{n!}{d_0!\, d_1!\, d_2!\, d_3!} \tag{3.15}$$

where the sum is over all sets of numbers $d_0, \dots, d_3$ such that d_0 is odd and $d_0 + d_1 + d_2 + d_3 = n$. On looking for a possible generating function, the factorials on the denominator lead one to try the exponential function $\exp(x)$ introduced in Example 2.10. So consider

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right).0$$

$$\times\left(1 + x + \frac{x^2}{2!} + \cdots\right)\text{………}(3.16)$$

The coefficient of x^n is $\frac{1}{n!}$ times the number given by (4.15), as can be seen by considering the ways in which x^n can be obtained by selecting a term from each bracket and multiplying them together. But (4.16) is

$$\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\{\exp(x)\}^3=\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\exp(3x)$$

$$=\frac{1}{2}(\exp(x)-\exp(-x))\exp(3x)$$

$$=\frac{1}{2}(\exp(4x)-\exp(2x))$$

The coefficient of x^n in this is

$$\frac{1}{2}\left(\frac{4^n}{n!}-\frac{2^n}{n!}\right)$$

The number of sequences is $n!$ times this number, namely

$$\frac{1}{2}(4^n-2^n)$$

For another method of solving this example, see Exercises 3, question 4. As far as the above solution is concerned, the reader should remember not so much the answer as the idea of making use of the properties of $\exp(x)$.

Example 4.8:

Partitions of an integer. Ideas from number theory have the habit of appearing all over the place and when least expected. One such idea is that of a partition of an integer. By a partition of a positive integer n is meant the expression of n as a sum of positive integers. For example, 5 has seven partitions:

$$5=4+1=3+2=3+1+1=2+2+1$$
$$=2+1+1+1=1+1+1+1+1$$

Note that 5 itself is a partition of 5. Let $p(n)$ denote the number of partitions of n, so that $p(5)=7$, and let $f(x)$ be the generating function,

$$f(x)=p(1)x+p(2)x^2+\cdots+p(n)x^n+\cdots$$

Consider the expression

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \ldots$$
$$= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\ldots.$$

What is the coefficient of x^n in this expression? Note that nx s can be obtained by selecting a power of x from the first bracket, another from the second, and so on, and multiplying them together. Thus if x^{ia_i} is chosen from the i th bracket, x^n will be obtained if n is the sum of $a_1 1$ s, $a_2 2$ s, and so on. Thus x^n will be obtained as many times as n has different partitions, so that the coefficient of x^n must be $p(n)$. This proves that the generating function is

$$f(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \ldots.$$

Although such a generating function does not yield a formula for $p(n)$ easily, it turns out to be useful enough to yield properties of partitions. For an example of this, see the exercises below.

Exercises 3:

1. Let $f(x)$ denote the generating function of the Fibonacci numbers. Show that the recurrence relation gives

$$f(x) = x + 2x^2 + x(f(x) - x) + x^2 f(x)$$

so that $(1-x-x^2)f(x) = x + x^2$

Deduce that

$$f(x) = (x+x^2)\{1-(x+x^2)\}^{-1}$$
$$= (x+x^2)\{1 + x(1+x) + x^2(1+x)^2 + \cdots\}$$

Read off the coefficient of x^n in this expression, and check that your answer agrees with Exercises 2, question 4.

2. Suppose that n objects lie in a straight line. Two adjacent objects are chosen and bracketed together, and thereafter are considered as just one object. This results in $(n-1)$ objects in a line. Two of these $(n-1)$ objects which are

adjacent are then bracketed together and thereafter considered as just one object. This process is continued until only one object is left. Let a_n denote the number of ways the process can be carried out, starting with n objects, so that $a_1 = 1$, $a_2 = 1, a_3 = 2$. By observing that in the last bracketing there are grouped together r original objects and ($n - r$) original objects, for some r, show that

$$a_n = a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1 \ (n \geqslant 3).$$

Deduce that the generating function $f(x)$ satisfies

$$\{f(x)\}^2 - f(x) + x = 0,$$

and hence show that -

$$a_n = \frac{(2n-2)!}{n!\,(n-1)!}.$$

3. Solve $a_n = 6a_{n-1} - 9a_{n-2}$ subject to the initial conditions $a_n = 2$, $a_1 = 6$ by writing $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and showing that $f(x) = 2(1-3x)^{-1}$.
4. Solve Example 4.7(b) as follows. Let a_n be the required number of n-digit sequences. By considering whether or not a given sequence begins with a 0 , show that

$$a_{n+1} = 3a_n + (4^n - a_n), \text{ i.e. } a_{n+1} = 2a_n + 4^n.$$

Put $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and show that

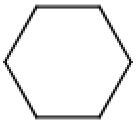
$$f(x) = \frac{x}{(1-2x)(1-4x)}$$

whence $a_n = \frac{1}{2}(4^n - 2^n)$.

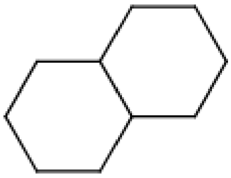
5. Let b_n denote the number of ways in which the sum n can be obtained on rolling a die any number of times. Show that the generating function for the b_i is

$$(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}.$$

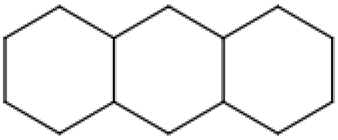
6. (Harary and Read (1970). Proc. Edinburgh math. Soc.). Certain organic chemical compounds built up from benzene rings can be represented by hexagons joined together:



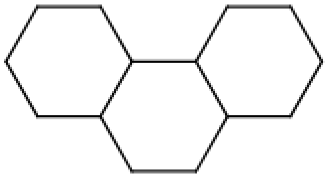
Benzene



Naphthalene

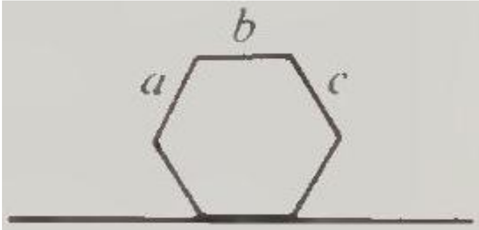


Anthracene



Phenanthracine

This raises the question: how many ways are there of combining together n hexagons? Simplify the problem as follows. First do not allow three hexagons to have a vertex in common. This means, for example, that a third hexagon cannot be nestled under two of the anthracene hexagons. Secondly, suppose that the configurations are all growing from a fixed spot, so that there is one fixed base hexagon. Onto this hexagon can be fitted either one hexagon (on any of the sides a, b, c) or two hexagons (one on each of a and c).



Let h_n denote the total number of possible patterns with n hexagons. Let s_n, d_n denote respectively the number with one, two hexagons joined to the base hexagon. Show that

(a) $s_n + d_n = h_n (n \geqslant 2)$,

(b) $s_{n+1} = 3h_n$,

(c) $d_{n+1} = h_1 h_{n-1} + h_2 h_{n-2} + \cdots + h_{n-1} h_1$.

If $h(x), s(x), d(x)$ are the respective generating functions, deduce that

(d) $h(x) = s(x) + d(x) + x$,

(e) $s(x) = 3xh(x)$,

(f) $d(x) = x\{h(x)\}^2$

and that

$$x\{h(x)\}^2 + (3x - 1)h(x) + x = 0.$$

7. Let $q(n)$ denote the number of partitions of n into distinct parts. Thus $q(5) = 3$, since 5 can be written as 5 or (4 + 1) or (3 + 2). Show that the generating function $Q(x)$ is

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \cdots$$

8. Let $r(n)$ denote the number of partitions of n into odd parts. Thus $r(5) = 3$ since $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Show that the generating function $R(x)$ is

$$(1 - x)^{-1}(1 - x^3)^{-1}(1 - x^5)^{-1} \cdots$$

9. Prove the surprising result that, in the notation of the previous exercises, $q(n) = r(n)$ for every value of n. Do this, without finding what $q(n)$ or

$r(n)$ is, by showing that the generating functions are the same. (Hint: in $Q(x)$ write $1+x^r$ as $\frac{1-x^{2r}}{1-x^r}$ and see what happens.)

10. Let $f(x)$ be the generating function for the sequence $a_1, a_2, \ldots$. Find the sequence whose generating function is $(1-x)f(x)$. The answer should explain why ($1-x$) is called the difference operator.

11. The sequence $\{a_n\}$ is defined by $a_0 = e, a_1 = 2e$,

$$na_n = 2(a_{n-1} + a_{n-2}), (n \geqslant 2)$$

Show that the generating function f satisfies the equation $f'(x) = 2(1+x)f(x)$ and deduce that $f(x) = \exp\{(1+x)^2\}$. Hence show that

$$a_{2n} = \sum_{r=0}^{\infty} \frac{1}{(n+r)!}\binom{2n+2r}{2n},$$

$$a_{2n+1} = \sum_{r=0}^{\infty} \frac{1}{(n+r+1)!}\cdot\binom{2n+2r+2}{2n+1}.$$

12. Let a_n denote the number of ways in which n letters can be selected from the alphabet $\{0,1,2\}$ with unlimited repetitions except that the letter 0 must be selected an even number of times. Find a_n. How many n-letter sequences can be formed from this alphabet containing an even number of $0s$?

4.4. Miscellaneous methods

The first recurrence relation mentioned in this book was

$$f(n,k) = f(n-1,k) + f(n,k-1) \tag{3.17}$$

subject to the boundary conditions

$$f(1,k) = 1, f(n,1) = n \tag{3.18}$$

This has certain similarities to the recurrence relation for binomial coefficients,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{3.19}$$

subject to the boundary conditions

$$\binom{n}{0} = 1, \binom{n}{n} = 1$$

No general method will be given for dealing with relations such as (3.17), but it will be shown how (3.17) can be solved by exploiting its similarity to the known relation (3.19). In (3.17) the k terms behave as in (3.19), but the n terms do not. Diagrammatically, the pattern is

$$\begin{matrix} n & n-1 & n \\ k & k & k-1 \end{matrix}$$

compared with the binomial coefficient pattern of

$$\begin{matrix} n & n-1 & n-1 \\ k & k & k-1 \end{matrix}$$

How can (3.17) be fitted into the required shape? Suppose the new function g is defined by

$$f(n,k) = g(n+k,k).$$

Then (3.17) becomes

$$g(n+k,k) = g(n+k-1,k) + g(n+k-1,k-1) \tag{3.20}$$

subject to the boundary conditions

$$g(1+k,k) = 1, g(n+1,1) = n$$

Now (3.20) is familiar. For, with $m = n+k$, it is just

$$g(m,k) = g(m-1,k) + g(m-1,k-1)$$

the recurrence relation for binomial coefficients. The boundary conditions, however, are not quite right. They would be, though, if the first variable were reduced by 1. So instead of putting $m = n + k$, try putting $u = n + k - 1$, and defining the function h by

$$f(n,k) = h(n+k-1,k) = h(u,k).$$

(3.17) now becomes

$$h(u,k) = h(u-1,k) + h(u-1,k-1)$$

subject to

$$h(k,k) = 1, h(n,1) = n.$$

It therefore follows that $h(u,k)$ must be $\binom{u}{k}$ so that, finally,

$$f(n,k) = \binom{n+k-1}{k}$$

Derangements

It has already been shown that if a_n denotes the number of derangements of n objects, then a_n satisfies the recurrence relation

$$a_n = (n-1)a_{n-1} + (n-1)a_{n-2} \tag{3.21}$$

This is not one of the relations covered by Theorem 4.1 since the coefficients of a_{n-1} and a_{n-2} are not constants but depend on n. How can (4.21) be solved? One idea is to make a suitable substitution which will transform (4.21) into something more tractable. Define a new sequence $\{b_n\}$ by writing

$$a_n = n!\, b_n$$

(3.21) then becomes

$$nb_n = (n-1)b_{n-1} + b_{n-2} \; (n \geqslant 3)$$

and the boundary conditions $a_1 = 0, a_2 = 1$, become

$$b_1 = 0, b_2 = \frac{1}{2}.$$

This new relation does not look much better than the original until it is observed that it can be written as

$$n(b_n - b_{n-1}) = -(b_{n-1} - b_{n-2})$$

which, on putting $c_n = b_n - b_{n-1}$, becomes

$$c_n = -\frac{1}{n} c_{n-1}, c_2 = \frac{1}{2}.$$

This is easily dealt with, for clearly

$$c_n = \frac{(-1)^n}{n!} \ (n \geqslant 2)$$

So that $b_n = c_n + b_{n-1}$

$= c_n + (c_{n-1} + b_{n-2}) = = c_n + c_{n-1} + b_{n-2}$

.....

.....

.....

$= c_n + c_{n-1} + \cdots \dots + c_2 + b_1$

$$= \sum_{r=2}^{n} \frac{(-1)^r}{r!}$$

Thus

$$a_n = n! \sum_{r=2}^{n} \frac{(-1)^r}{r!} = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

$$a_n = n! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \dots + (-1)^n \frac{1}{n!}\right\} \quad \dots\dots\dots\dots(3.22)$$

This formula for a_n will be derived by another method when the inclusion-exclusion principle is introduced in the next chapter.

Exercises 4:

1.Let a_n denote the number of derangements of n objects. Deduce from (3.21) that a, =na,, + (— 1)", and hence derive (3.22) by con- a side ring $b_n = \frac{a_n}{n!}$ Also verify that a_4 = 9, a_5 = 44, a_6 = 265, a_7 = 1854.

2. As in Exercises 1.1, question 5, let g(n, k) denote the number of ways of placing k indistinguishable lions in n cages so that no cage contains more than one lion and no two lions are put in consecutive cages. It has been shown that g(n,k)=g(n-2,k-1)+g(n-1,k) Define a new function h by g(n, k) =h(p, k) where p =n — k + 1. Show that the recurrence relation and boundary conditions reduce to those —k+1 for binomial coefficients, and deduce that $g(n,k) = \binom{n-k+1}{k}$.

Unit IV

The Inclusion-Exclusion Principles-Rook Polynomial.

Chapter 4: Sections 4.1 - 4.2

4.1. The Inclusion-Exclusion Principles:

The principle three are in mathematics a handful of principles which look so simple as to be valueless, but yet in practice are of the utmost importance and power. One such principle is the box principle which asserts that if (n + 1) lions are put into 7 cages, then at least one cage must contain more than one lion. A course in number theory will show how powerful this simple principle is. The principle which is the subject of the present chapter is not much more difficult to understand. In-its simplest form it is concerned with the number of elements in the union of two sets A and B (see Fig. 4.1).

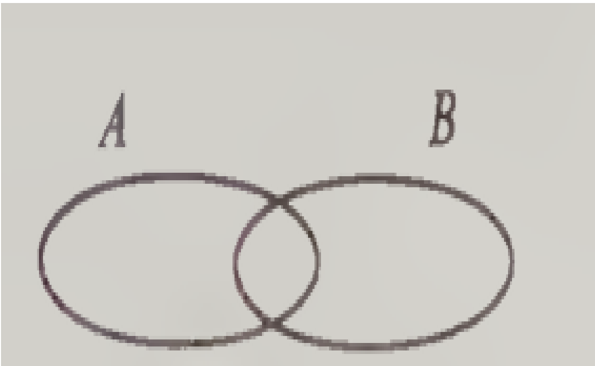


Figure 4.1.

Let |A | denote the number of elements in the set A. In evaluating |A U B|, consider first the possible answer |A| + |B]. This will in general be the wrong answer since those elements which are in both A and B are included twice, and therefore must be removed once.

Thus

$|A \cup B| = |A| + |B| - |A \cap B|$ ……..(4.1)

Here the inclusion-exclusion principle is at work. First too many are included, but thereafter excluded.

Example 4.1:

$A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. $|A| = |B| = 3$, $|A \cap B| = 2$, so that $|A \cup B| = 3 + 3 - 2 = 4$. This is correct since $A \cup B = \{1,2,3,4\}$

What happens with 3 sets A, B, C (see Fig. 4.2)? In evaluating $|A \cup B \cup C|$, start off with $|A|+|B|+|C|$. Any element in both A and B, or in both B and C, or in both C and A is included more than once. So the next attempt at a solution to consider is

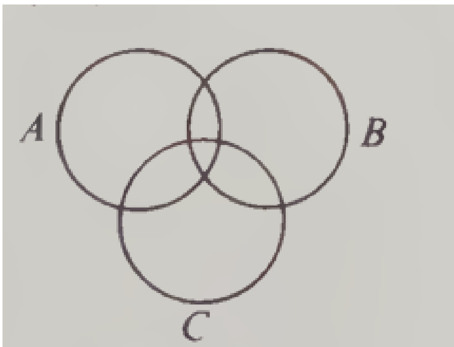


Figure 4.2.

Not even this is correct, for if there are any elements in all three sets A, B,C, then they will have been included thrice and excluded thrice, and so must be added in once again. Thus

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$ ……….(4.1)

The reader will now be able to deduce from the patterns in (4.1) and (4.2) a similar expression for $|A \cup B \cup C \cup D|$. Indeed,

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |B \cap C| - |C \cap D| - |A \cap C| - |B \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

All these expressions illustrate the basic inclusion-exclusion principle, which is now presented in a slightly different way. Suppose that a collection of objects is given, along with a list of r properties which the objects may or may not possess, and suppose that it is required to find the number of objects which possess at least one of the properties. In the examples above, the first property was that of belonging to the set A, the second was that of belonging to B, and so on. Denote by N(i,j, . . ., k) the number of objects which possess each of the ith, jth, ..., kth properties (and possibly some others as well). Then the number of objects possessing at least one of the properties is

N(1)+N(2)+N(3)+……+N(r) –{N(1,2)+N(1,3)+N(2,3)+……+N(r-1,r)}+
{N(1,2,3)+N(1,2,4)+……..+N(r-2,r-1,r) }-
$$\ldots\ldots\ldots+(-1)^{r-1}N(1,2,\ldots r) \qquad \ldots..(4.3)$$

This result is perhaps more useful in its complementary form. Instead of asking how many objects possess at least one of the properties, it is asked how many possess none of the properties. Clearly this is obtained by sub- tracting the expression (4.3) from the total number of objects.

Proof of the principle (4.3). If an object possesses none of the r properties, then it clearly contributes nothing to (4.3). If an object possesses $t \geqslant 1$ properties, it must be shown that it contributes 1 to (4.3). But its contribution is

$$t-\binom{t}{2}+\binom{t}{3}-\binom{t}{4}+\cdots$$
$$=1-\left\{1-t+\binom{t}{2}-\binom{t}{3}+\cdots\right\}$$
$$=1-(1-1)^t=1$$

Example 4.2:

Derangements. The formula (4.22) for a_n, the number of derangements of n symbols, has the inclusion-exclusion look about it. Its appearance suggests that an alternative derivation is possible, and this is now confirmed. As objects, take the $n!$ possible permutations of the n symbols. An object possesses the i th property if the i th symbol appears in it in the i th place. Then the number of derangements is just the number of objects possessing none of the properties. Using the notation of (4.3),

$$N(i) = (n-1)!$$

since the i th symbol is fixed and the remaining ($n-1$) can undergo any permutation. Similarly,

$$N(i,j) = (n-2)!,$$

since two symbols are fixed, leaving $n-2$ to be permuted;

$$N(i,j,k) = (n-3)!$$

and so on. Further, the number of terms of type $N(i)$ is $\binom{n}{1}$, of type $N(i,j)$ is $\binom{n}{2}$, and so on. Thus, the number of permutations which satisfy at least one of the properties, i.e. which are not derangements, is

$$(n-1)!\binom{n}{1} - (n-2)!\binom{n}{2} + (n-3)!\binom{n}{3} - \cdots$$

The number of derangements is $n!$ minus this number, i.e.

$$n! - \frac{(n-1)!\,n!}{(n-1)!\,1!} + \frac{(n-2)!\,n!}{(n-2)!\,2!} - \frac{(n-3)!\,n!}{(n-3)!\,3!} + \cdots$$
$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right\}$$

It so happens that it is profitable to consider this problem geometrically. Take an $n \times n$ chessboard (see Fig. 4.3), and represent a permutation of the numbers $1,2,\ldots,n$ by placing a chesspiece on the square of the i th row and the j th column if the number i is permuted to the j th position. For example, the permutation 2413 is represented by the accompanying diagram (where the top row is taken as the first row, and the left column as the first column).

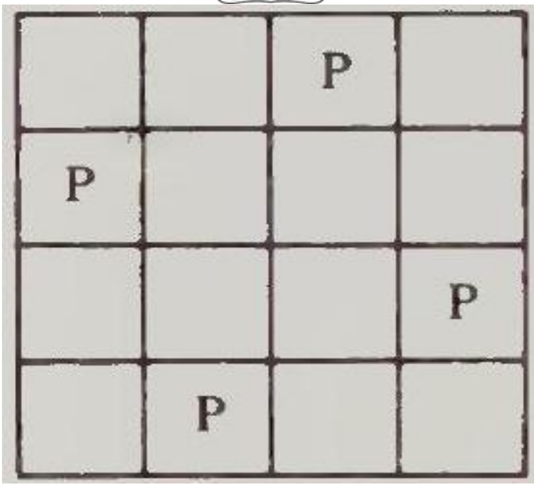


Figure. 4.3

Clearly a permutation corresponds to placing n pieces on an $n \times n$ board so that no two pieces lie in the same row or column. For a derangement, no piece must lie on the main diagonal (i.e. the diagonal from the top left to the bottom right). Thus formula (3.22) can be interpreted as giving the number of ways of placing n rooks on an $n \times n$ chessboard, with none on the main diagonal, so that no rook can take any other rook. For, as is well known, a rook can only move along rows or columns.

This idea will be returned to later, but meanwhile another interpretation of (3.22) is given. Suppose that in constructing an $n \times n$ Latin square the numbers $1,2, \dots, n$ have been placed in some order in the first row. Then (3.22) gives the number of ways of choosing a second row for the square. This raises the question: assuming that the first ($r-1$) rows have been chosen, can anything be said about the number of choices for the r th row? This is clearly closely related to the enumeration of permutations of $1, \dots, n$ where there are ($r-1$) forbidden places for each number.

Exercises 5.1

1. Exam scripts of n students are returned to the class at random, one to each student. Show that the probability that no student receives his own script tends to $1/e$ as $n \to \infty$. (Probability = number of ways this can happen divided by the total number of ways in which the scripts can be returned.)
2. Each of a class of 50 students reads at least one of mathematics and physics. 30 read mathematics and 27 read both. How many read physics?
3. How many integers from 1 to 1000 are divisible by none of $3,7,11$?
4. A survey carried out over a large number of citizens of a certain city revealed that 90 per cent of all people detest at least one of the pop stars Hairy, Dirty, and Screamer. 45 per cent detest Hairy, 28 per cent detest Dirty, and 46 per cent detest Screamer. If 27 per cent detest only Screamer, and 6 per cent detest all three, how many detest Hairy and Dirty but not Screamer?
5. Present the permutation 35142 by a chessboard diagram.
6. How many ways are there of placing 5 non-taking rooks on a 5×5 board? How many ways if none lie on the main diagonal? How many if exactly one lies on the main diagonal?
7. How many permutations are there of the digits $1,2, \ldots ,8$ in which none of the patterns $12,34,56,78$ appears?

4.2. Rook polynomials:

It has already been pointed out that the problem of derangements is equivalent to that of placing non-taking rooks on certain allowable squares of the chessboard. This suggests that some combinatorial problems may reduce to placing non-taking rooks on boards of various shapes and sizes.

Let C be an arbitrary board of any shape, with m squares. For each $k \leqslant m$, let $r_k(C)$ denote the number of ways of placing k non-taking rooks on C. Then the generating function for the numbers $r_k(C)$,

$$R(x, C) = r_0(C) + r_1(C)x + r_2(C)x^2 + \cdots + r_m(C)x^m,$$

is called the rook polynomial of the board C.

Example 4.3:

Find the rook polynomial for an ordinary 4×4 board.

Solution:

The numbers $r_i(C), i = 0, \ldots, 16$ must be evaluated.

Clearly $r_i(C) = 0$ for all $i > 4$.

$r_0(C) =$ number of ways of placing no non-taking rooks on $C, = 1$

(the only way being to leave the board empty).

$r_1(C) = 16$, since there are 16 squares to choose from. Next, $r_2(C)$ is the number of ways of placing two non-taking rooks on C. These rooks must lie in different rows and columns. The number of ways of choosing two rows in $\binom{4}{2}$.

Once the rows are chosen, a rook can be placed in the first one in any of 4 ways, and another rook in the second row in any of 3 places.

Thus

$$r_2(C) = \binom{4}{2} \cdot 4 \cdot 3 = 72. \text{ Similarly}$$

$$r_3(C) = \binom{4}{3} \cdot 4 \cdot 3 \cdot 2 = 96, r_4(C) = \binom{4}{4} \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Thus

$$R(x, C) = 1 + 16x + 72x^2 + 96x^3 + 24x^4$$

The reader should now try Exercises 4.2, question 1. Faced with a more awkwardly shaped board, the problem of finding the rook polynomial would prove to be near impossible if it were not that some tricks exist whereby a board can be reduced to a simpler one. One such trick is concerned with boards which fall into two or more noninterfering parts. Two parts A, B of a chessboard C are non-interfering if no square in A is in the same row or column of C as any square of B. The board in Fig. 4.4 falls into 3 non-interfering parts.

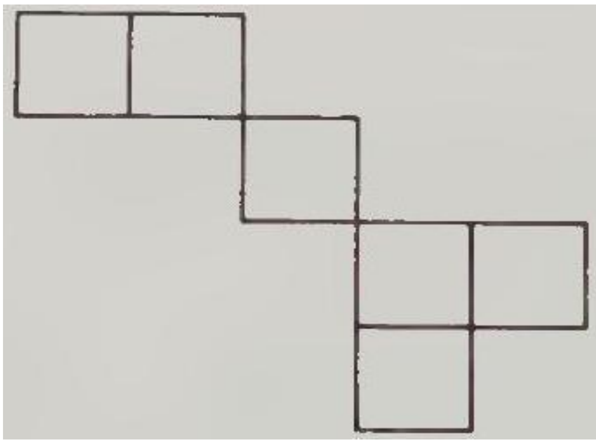


Figure. 4.4

Property 1:

If a chessboard C consists of two non-interfering parts, then the rook polynomial for C is just the product of the rook polynomials for the parts A and B.

Proof:

When k non-taking rooks are placed on C, t will be placed on A and ($k - t$) on B, for some $t, 0 \leqslant t \leqslant k$. Since the $r_t(A)$ possible placings of t rooks on A can each occur along with any of the $r_{k-t}(B)$ placings on B (for A and B do not interfere with one another), it follows that

$$r_k(C) = r_0(A)r_k(B) + r_1(A)r_{k-1}(B) + \cdots + r_k(A)r_0(B).$$

But the expression on the right is simply the coefficient of x^k in

$$\{r_0(A) + r_1(A)x + r_2(A)x^2 + \cdots\}\{r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots\},$$

i.e. in the product of the rook polynomials for A and B.

Example 4.4:

Suppose that C consists of n non-interfering 2×2 blocks (Fig. 4.5).

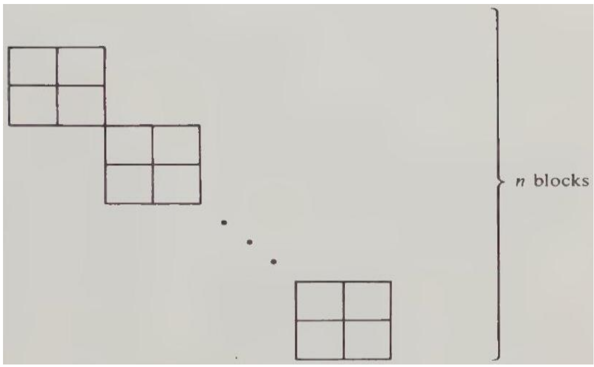


Figure. 4.5

The rook polynomial for one block is $1 + 4x + 2x^2$. The rook polynomial for C is therefore $(1 + 4x + 2x^2)^n$.

Property 1, although useful, is not widely applicable. The problem still remains of how to deal with a board which does not fall into noninterfering parts. The next property is of use here.

Property 2:

Given a chessboard C, choose any square of C and let D denote the board obtained by deleting from C every square in the same row or column as the chosen square (including the chosen square itself).

Let E denote the board obtained from C by deleting only the chosen square. Then

$$R(x, C) = xR(x, D) + R(x, E).$$

Proof:

If $k \geqslant 1$ non-taking rooks are placed on C, then the chosen square either is or is not used. If it is used, then ($k-1$) rooks are left to be placed on D, and this can be done in $r_{k-1}(D)$ ways. If it is not used, then k rooks have to be placed on E, and this can be done in $r_k(E)$ ways. Thus

$$r_k(C) = r_{k-1}(D) + r_k(E),$$

so that

$$\begin{aligned} R(x,C) &= \sum_{k=0}^{\infty} r_k(C)x^k \\ &= \sum_{k=1}^{\infty} r_{k-1}(D)x^k + \sum_{k=0}^{\infty} r_k(E)x^k \\ &= xR(x,D) + R(x,E) \end{aligned}$$

By repeated applications of Property 2, the rook polynomial of any board can be found.

Example 4.5:

Find the rook polynomial of the board of Fig. 4.6.

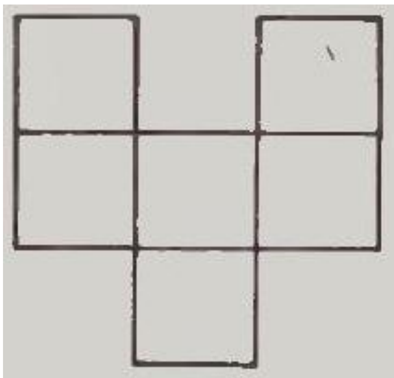


Figure. 4.6

Solution:

Choosing the centre square,

$$R(x,C) = xR(x,D) + R(x,E), \dots\dots (4.4)$$

where

$$D = \square\square \text{ and } E = \square\square =$$

Now,

$$R(x, D) = 1 + 2x \tag{4.5}$$

and, by Property 1,

$$R(x, E) = R(x, H)R(x, K)$$

where

$$H = \square\square \quad \text{and} \quad K = \square$$

Since $R(x, H) = 1 + 4x + 2x^2$ and $R(x, K) = 1 + x$, it follows that

$$R(x, E) = (1 + x)(1 + 4x + 2x^2) \tag{4.6}$$

From (4.4)-(.6) it now follows that

$$\begin{aligned} R(x, C) &= x(1 + 2x) + (1 + x)(1 + 4x + 2x^2) \\ &= 1 + 6x + 8x^2 + 2x^3 \end{aligned}$$

This whole argument can be written more clearly in the following symbolic way:

$$\begin{aligned} R(x, C) &= xR(\square\square) + R(\square\square) \\ &= x(1 + 2x) + R(\square\square)R(\square) \\ &= x(1 + 2x) + (1 + 4x + 2x^2)(1 + x) \end{aligned}$$

This abbreviated notation is used in the next example.

Example 5.6. $R(\square\square) = xR(\square) + R(\square\square)$

$$\begin{aligned} &= x(1 + x) + xR\begin{pmatrix}\square \\ \square\end{pmatrix} + R\begin{pmatrix}\square \\ \square\end{pmatrix} \\ &= x(1 + x) + x(1 + 2x) + R\begin{pmatrix}\square \\ \square\end{pmatrix}R(\square) \\ &= x(2 + 3x) + (1 + 3x + x^2)(1 + x) \\ &= 1 + 6x + 7x^2 + x^3 \end{aligned}$$

Applications of rook polynomials

Example 5.7:

The manager of a firm has 5 employees to be assigned to 5 different jobs. The men are A, B, C, D, E and the jobs are a, b, c, d, e. He considers that A is unsuited for jobs b and c, B unsuited for a and c, C unsuited for b, d and e, D suited for all and E unsuited for d. In how many ways can he assign the jobs to men suited to them?

Solution:

The board shown in Fig. 5.7 represents the situation. The problem is to find the coefficient of x^5 in the rook polynomial for this board. At this point the reader will probably hold back at the mere thought of finding the rook polynomial, due to the amount of work

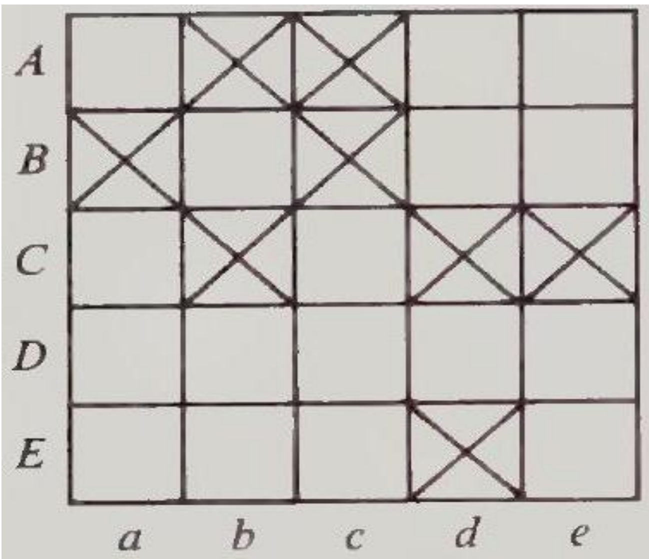
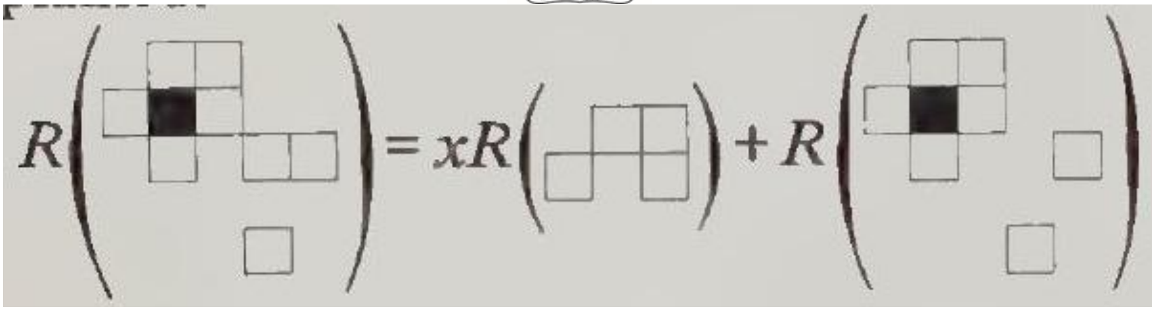


Figure. 4.7

involved. In fact, it would be much easier to find the rook polynomial for the board consisting of the forbidden positions. This polynomial will now be found before its usefulness to the original problem is explained.



$= x\{xR(□) + R(□)\} + R(□)R(□$

$= x\{x(1+x) + 1 + 3x + 2x^2\} + (1+x)\{xR(□□) + R(□□)\}$

$= x(1 + 4x + 3x^2) + (1+x)\{x(1 + 3x + x^2) + + (1+x)R(□□)\}$

$= (x + 4x^2 + 3x^3) + (1+x)\{x + 3x^2 + x^3 + + (1+x)(1 + 4x + 3x^2)\}$

$= 1 + 8x + 20x^2 + 17x^3 + 4x^4$.(4.7)

This is the rook polynomial for the board consisting of the forbidden squares. Now the assignment of jobs to men can be considered as permutations of the numbers $1, \ldots, 5$. For example, if A gets job c, B gets d, C gets b, D gets a, and E gets e, the assignment corresponds to the permutation 34215, since, for example, the first man gets the third job and the fourth man gets the first job. The key to the problem now lies in the following theorem.

Theorem 4.1:

The number of permutations of n symbols in which no symbol is in a forbidden position is

$$\sum_{k=0}^{n} (-1)^k (n-k)!\, r_k$$

where r_k is the number of ways of placing k non-taking rooks on the board of forbidden positions.

Solution to Example 4.7 (continued). Assuming for the moment that the theorem has been proved, and noting that, from (4.7),

$$r_0 = 1, r_1 = 8, r_2 = 20, r_3 = 17, r_4 = 4,$$

and $n = 5$, the number of ways of assigning the jobs to the men is

$$5! - 4!\,8 + 3!\,20 - 2!\,17 + 4 = 18$$

Thus,

a knowledge of the rook polynomial for the board of forbidden squares leads very quickly to information about the permitted squares.

Proof of Theorem 5.1. In the notation of the inclusion-exclusion principle, suppose that a permutation possesses the i th property if the i th symbol is in a forbidden position. Then the number of permutations with no symbol in a forbidden position is

$$n! - \{N(1) + \cdots + N(n)\} + \{N(1,2) + \cdots\} - \cdots$$

Now each $N(i)$ is equal to $s_i(n-1)\,!$ where s_i is the number of forbidden squares in the i th row, since the i th symbol can be placed on any of these s_i squares and the remaining symbols can be placed in $(n-1)\,!$ ways. Since $s_1 + \cdots + s_n = r_1$, it follows that

$$N(1) + \cdots + N(n) = (n-1)!\,(s_1 + \cdots + s_n) = (n-1)!\,r_1.$$

Similarly,

$$N(1,2) + N(1,3) + \cdots + N(n-1,n) = (n-2)!\,r_2$$

and so on.

Example 4.8:

In constructing a 6×6 Latin square, the first two rows have been chosen as follows:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5. \end{array}$$

By Hall's theorem (2.3) it is definitely possible to find a suitable third row. But how many possibilities are there?

Solution:

The problem is: how many permutations of $1, \ldots, 6$ are there with no symbol in a forbidden position, the forbidden positions being represented by crosses in the diagram (Fig. 4.8)?

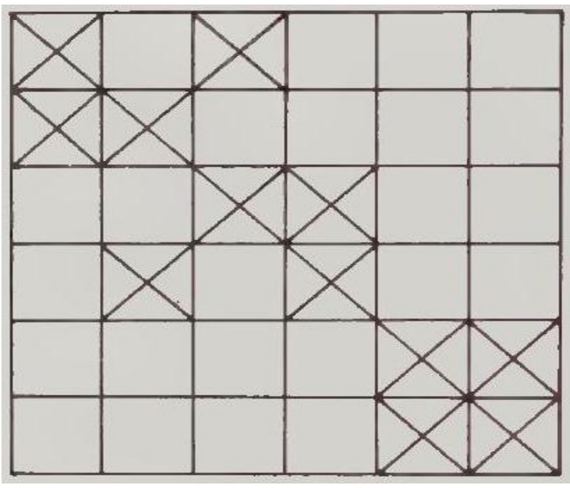


Figure. 4.8

Following the method of the previous example, the first thing to do is to obtain the rook polynomial for the board of forbidden positions. This is

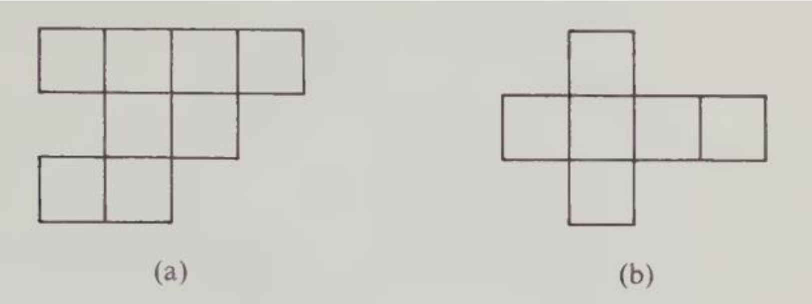
$$R(\square\square)R(\square) = (2x^4 + 16x^3 + 20x^2 + 8x + 1)(1 + 4x + 2x^2)$$
$$=(4x^6 + 40x^5 + 106x^4 + 112x^3 + 54x^2 + 12x + 1)$$

In the notation of Theorem 4.1, $r_6 = 4, r_5 = 40, r_4 = 106, r_3 = 112, r_2 = 54, r_1 = 12$, and $r_0 = 1$, so that the number of possibilities for the third row of the square is

$$6! - 12.5! + 54.4! - 112.3! + 106.2! - 40 + 4 = 70.$$

Exercises 5.2

1. Find the rook polynomial for an ordinary 8×8 chessboard.
2. A six-a-side football team is to consist bf the players 1? A,B,...,F. A refuses to play in positions 1 or 2, B in position 4, C in positions 1 or 5, D in 2, — in 4, and F in 4 or 6. How many ways are there of assigning agreeable positions to the six players?
3. The first two rows of a 5 x 5 Latin square are 1, 2, 3, 4, 5 and 2, 3, 4,5, 1. In how many ways can a third row be chosen?
4. Find the rook polynomials for the following boards:



5. A computer matching service has five male subscribers A, B,C, D, E and four female subscribers a, b, c,d. After analysing their interests and personalities, the computer decides that a is unsuitable for B and C, 6 unsuitable for C, c forA and £, d for B. In how many ways can the female subscribers be matched?
6. By trial and error, verify that there are four possible third rows for a Latin square whose first two rows are 1, 2, 3, 4 and 2, 1, 4, 3, whereas there are only two possibilities if the first two rows are 1, 2, 3, 4 and 2, 4, 1, 3. Thus, although the number of choices for the next row is always => 1, the actual number of choices depends on the choice of the previous rows.

7. In the light of question 8, repeat question 4 if the second row is 2,3,1,5,4.

Unit V

Block designs - square block designs.

Chapter 5: Sections 5.1, 5.2

6.1. Block designs

The origins of the theory of block designs can be traced back to the problem of designing certain types of statistical experiment. It is therefore not insignificant that the name of the distinguished statistician Fisher is attached to one of the first results in the subject (Theorem 5

.2). The idea behind a block design can be seen in the following type of problem. Suppose that a number of brands of instant coffee are to be tested among a number of housewives, the object of the experiment being to let the ladies compare the different brands and decide on their relative merits. To make the tests as fair as possible, it is decided that the following conditions should be satisfied: (1) each housewife should taste the same number of brands; (2) each pair of brands should be compared by the same number of housewives. % Clearly, one way of achieving this would be to give every housewife every brand of coffee, but this is wasteful and time consuming. The problem is to achieve the aim more economically. Mathematically, all that is involved is a set S of varieties (the brands of coffee), and a collection of subsets of S (each subset consisting of those varieties which a particular housewife tastes) called blocks, with the properties: (a) each block has the same number of elements; (b) every pair of varieties is contained in the same number of blocks.

Definition 5.1.

A block design is a family of b subsets of a set S of v elements such that, for some fixed k and A, with k < 0, (1) each subset has k elements, (2) each pair of elements of S occur together in exactly \ subsets. The elements of S are called the varieties, and the subsets of S are called the blocks.

Example 6.1. Take S = {1, 2, ..., 7}, and consider the following seven subsets of S: {1,2,4}, {2,3,5}, {3,4,6}, {4,5,7}, {5,6,1}, {6,7,2}, {7,1,3}.Here b=7,v=7,k=3, $\lambda = 1$.To see that λ = 1, consider any pair of elements, say 4 and 6, and verify that exactly one of the seven subsets contains both 4 and 6. Do this for each pair.

This design could be used to compare 7 brands of coffee, using seven housewives. Each housewife is given 3 brands, and any particular pair of brands will be compared by exactly one housewife.

There is a simple geometrical representation of the above design. The elements $1, \ldots, 7$ are represented by points, and the blocks are represented by lines (all but one being a straight line). This is the simplest example of a finite projective plane, where the elements are usually called points and the blocks are called lines. This one is known as the seven-point plane (Fig 5.1). It is the simplest example of a Steiner triple system;

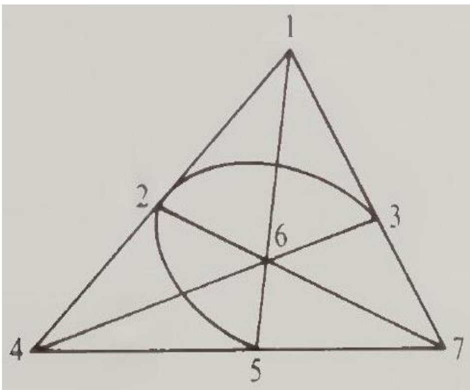


Figure. 5.1. The seven-point plane.

But there is another useful way of representing the design of Example 5.1. The first set {1,2,4} can be represented by the following string of 0s and 1 s:

1101000.

There is a 1 in the first, second, and fourth places because the set consists of the first, second, and fourth elements. Similarly, $\{2,3,5\}$ can be represented by

$$0110100.$$

Representing each set in this way, and listing the strings one under the other, the following matrix, called the incidence matrix of the design, is formed

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Each row represents a subset or block, and each column gives information about a particular element or variety. For example, by looking at the third column it is deduced that the element 3 occurs in the second, third, and seventh sets. The condition that any pair of elements occur together in exactly one block is represented by the property that any two columns both have 1s in the same row exactly once. For example, the first and seventh columns both have a 1 in the seventh row; this means that the elements 1 and 7 occur together only in the seventh set.

The advantage of using incidence matrices to describe a block design instead of listing the sets element by element is that the structure of the design is seen more clearly without any irrelevant information such as the names of the elements confusing the issue. It is also easier to scan the columns to find how many sets contain a given element than to look through a list of sets. Note also that the number of rows is b, and the number of columns is v.

The conditions for a block design imply a further condition, namely that each variety must occur in the same number of blocks. The proof of this acts as an introduction to simple but important counting ideas which will be much used in the next chapter.

Theorem 5.1:

In a block design each element lies in exactly r blocks, where

$$r(k-1) = \lambda(v-1) \text{ and } bk = vr. \tag{5.1}$$

Proof:

Concentrate on any one of the elements, and suppose that it occurs in r blocks, for some r. Each of these r blocks contain ($k-1$) other elements, so that the number of pairs including this chosen element is $r(k-1)$. But there are ($v-1$) elements with which it can be paired, and each pair occurs λ times. Hence $r(k-1) = \lambda(v-1)$. Since k, v, and λ are fixed, it follows that r must be the same for each element. For this fixed value of r, each element therefore has r appearances in the blocks, so that there are vr appearances of elements altogether. But there are b blocks each with k elements, so the number of appearances must also be bk. Thus $bk = rv$.

The five parameters b, v, r, k, λ of a block design are therefore not independent, but have two restrictions as stated in the theorem. Often a block design is referred to as a (b, v, r, k, λ)-configuration;

for example, the seven-point plane is a (7,7,3,3,1)-configuration. Whatever b, v, r, k, λ are, they must satisfy (5.1), but conversely, if five numbers b, v, r, k, λ satisfy (5.1), there is no guarantee that a (b, v, r, k, λ)-configuration exists. For example, it is known that a finite projective plane with $b = v = 43, r = k = 7$, and $\lambda = 1$ does not exist (see Theorem 5.4).

The seven-point plane has a further property which is not possessed by all block designs, namely that $b = v$. This means that the number of blocks is the same as the number of elements, so that the incidence matrix is a square matrix. Such designs are called square or symmetric designs although the second description is misleading since the incidence matrix need not be symmetric about the main diagonal. The reason for the name will appear later. Since $b=v$ implies $r=k$, square designs are completely determined by the three parameters v, k, λ and hence are often called (v, k, λ)-configurations. The seven-point plane is a (7,3,1)-configuration. Condition (6.1) becomes

$$k(k-1)=\lambda(v-1) \tag{5.2}$$

Equality of b and v is in a sense the extreme case since b can never be smaller than v in a block design. This is Fisher's result, proved in 1940.

Theorem 5.2 (Fisher). For $a(b, v, r, k, \lambda)$-configuration,

$$b \geqslant v.$$

Proof:

Let A be the incidence matrix, so that A has b rows and v columns. The key idea in the proof is to determine the matrix $C=A^{\prime} A$. Here $A^{\prime}$ is the transposed matrix of A, obtained by writing the rows of A as columns, and the columns as rows. The element $a_{i j}^{\prime}$ in the i th row and j th column of $A^{\prime}$ is equal to $a_{j i}$, the element in the j th row and the i th column of A. Then $C=\left(c_{i j}\right)_{v \times v}$,

where

$$\begin{aligned} c_{i j} & =\sum_h a_{i h}^{\prime} a_{h j} \\ & =\sum_h a_{h i} a_{h j} . \end{aligned}$$

In particular,

$$c_{i i}=\sum_h a_{h i}^2=\sum_h a_{h i}$$

since each a_{ij} is 0 or 1 and $0^2 = 0$ and $1^2 = 1$. But $a_{hi} = 1$ if and only if the i th element is in the h th set, and is 0 otherwise. Thus

$$c_{ii} = \sum_h a_{hi} = \text{ number of sets containing the } i \text{ th element}$$

$$= r.$$

Also, if $i \neq j$,

$$c_{ij} = \sum_h a_{hi}a_{hj}.$$

Now $a_{hi}a_{hj}$ is equal to 1 if and only if $a_{hi} = a_{hj} = 1$, i.e. only if the h th set contains both the i th and the j th elements. There are λ such h s. Thus $c_{ij} = \lambda$, and

$$C = A'A = \begin{bmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \vdots & & & & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{bmatrix}.$$

If I is used to denote the unit matrix with 1 s down the main diagonal and 0 s elsewhere, and J denotes the matrix with every entry equal to 1 , this result can be written as

$$A'A = (r - \lambda)I + \lambda J. \tag{5.3}$$

Exercise for the reader. If J is $v \times v$, then $J^2 = vJ$.

To prove that $b \geqslant v$, note first that, if $\rho(C)$ denotes the rank of the matrix C,

$$\rho(C) = \rho(A'A) \leqslant \rho(A) \leqslant b. \tag{5.4}$$

Use is being made here of the facts that $\rho(XY) \leqslant \rho(Y)$ for all matrices X, Y, and the rank of a matrix is no greater than the number of its rows or columns.

However,

$$\det\begin{bmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \vdots & & & & \\ \lambda & \lambda & \lambda & \dots & r \end{bmatrix} = \det\begin{bmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda - r & r - \lambda & 0 & \dots & 0 \\ \lambda - r & 0 & r - \lambda & \dots & 0 \\ \vdots & & & & \\ \lambda - r & 0 & 0 & \dots & r - \lambda \end{bmatrix}$$

on substracting the first row from each of the others, and this, in turn, on adding to the first column all the other columns, is equal to

$$\det\begin{bmatrix} r + \lambda(v-1) & \lambda & \lambda & \dots & \lambda \\ 0 & r - \lambda & 0 & \dots & 0 \\ 0 & 0 & r - \lambda & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & r - \lambda \end{bmatrix}$$

$$= \{r + (v-1)\lambda\} \cdot (r - \lambda)^{v-1}$$

$$= rk(r - \lambda)^{v-1}$$

$$\neq 0, \text{ since } (5.1) \text{ implies that } r > \lambda$$

Thus C is a nonsingular $v \times v$ matrix, and so $\rho(C) = v$. Thus (6.4) gives $b \geqslant v$ as required.

Exercises 1:

1.The following 12 sets form a (b, v, r, k, λ)-configuration.

$$\begin{matrix} \{1,2,3\} & \{4,5,6\} & \{7,8,9\} & \{1,4,7\} \\ \{2,5,8\} & \{3,6,9\} & \{1,5,9\} & \{2,6,7\} \\ \{3,4,8\} & \{1,6,8\} & \{2,4,9\} & \{3,5,7\} \end{matrix}$$

Write down the incidence matrix and check that $b = 12, v = 9, r = 4, k = 3, \lambda = 1$. Verify that $bk = rv$ and $r(k-1) = \lambda(v-1)$. Explain how this design could be used to test 9 detergents with the help of 12 housewives, or with the help of 3 housewives on 4 consecutive days.

2. Find $A'A$ and AA' for the seven-point plane.

3. Show that there exists no (12,8,3,2,1)-configuration.

4. Define the complement of a design D to be the design obtained by changing 0 to 1 and 1 to 0 throughout the incidence matrix of D. If D is a (b, v, r, k, λ)-

configuration, show that its complement is a ($b, v, b-r, v-k, b-2r+\lambda$)-configuration.

5. Derive from question 1 above a (12,9,8,6,5)-configuration.

6. Show that no block design exists with (a) $v=16, k=6, \lambda=1$, (b) $v=21, k=6, \lambda=1$, (c) $v=25, k=10, \lambda=3$, although in each case (6.1) is satisfied.

7. Show that there is essentially only one (7,7,3,3,1) design, as follows. Assume that the elements are $1,2, \ldots, 7$; by relabelling if necessary assume that $\{1,2,4\},\{2,3,5)$ and $\{1,5,6\}$ are blocks: show that the remaining blocks are uniquely determined.

8. One of the first block designs to appear explicitly in the statistical literature was the following, due to Yates [29] in 1936.

$$\{a, b, c\},\{a, b, d\},\{a, c, e\},\{a, d, f\},\{a, e, f\},\{b, c, f\}$$
$$\{b, d, e\},\{b, e, f\},\{c, d, e\},\{c, d, f\}$$

Verify that this is a (10,6,5,3,2) design. It could be used in an agricultural experiment where a research lab has 10 blocks each with 3 plots, and where there are 6 varieties of wheat, 3 in each block, arranged so that any two varieties can be compared twice due to their occurring twice in the same block.

9. One of the problems which stimulated interest in block designs in the nineteenth century was Kirkman's schoolgirls problem. The problem was to arrange 15 schoolgirls in 5 groups of 3 on each of the 7 days of a week in such a way that during the week each pair of girls would walk together exactly once. What was wanted was a (35, 15, 7, 3, 1) design which is resolvable, i.e. such that the blocks can be grouped into $r=7$ groups of 5 blocks so that each element occurs precisely once in each group. Let the 15 girls be labelled $X_1, \ldots, X_7, Y_1, \ldots, Y_7, Z$. Verify that the following is a solution to the problem; it has

the nice property that the triples for each day can be obtained from those of the previous day by replacing X_i by X_{i+1}, Y_i by $Y_{i+1}(i \leqslant 6), X_7$ by X_1, Y_7 by Y_1.

Day1: X_1Y_1Z $X_2X_6Y_4$ $X_3X_4Y_7$ $X_5X_7Y_6$ $Y_2Y_3Y_5$

Day 2: X_2Y_2Z $X_3X_7Y_5$ $X_4X_5Y_1$ $X_6X_1Y_7$ $Y_3Y_4Y_6$

………

Day 7: X_7Y_7Z $X_1X_5Y_3$ $X_2X_3Y_6$ $X_4X_6Y_5$ $Y_1Y_2Y_4$.

5.2. Square block designs:

In the special case of a square design, (6.3) becomes

$$A'A = (k-\lambda)I + \lambda J = \begin{bmatrix} k & \lambda & \lambda & \dots & \lambda \\ \lambda & k & \lambda & \dots & \lambda \\ \vdots & & & & \vdots \\ \lambda & \lambda & \lambda & \dots & k \end{bmatrix}. \tag{5.5}$$

As has been pointed out already, the incidence matrix A of a (v, k, λ) configuration need not itself be symmetric. The reason for calling a square block design symmetric is that there is the following symmetry in the properties of the rows and columns of the incidence matrix:

(1) Any row contains k 1s.

(2) Any column contains k 1s.

(3) Any pair of columns both have 1 s in exactly λ rows.

(4) Any pair of rows both have 1 s in exactly λ columns.

Property (4) has not yet been proved, but it will be shown to follow from (1), (2) and (3). Note that property (4) says: in a symmetric (v, k, λ) configuration each pair of blocks interesect in exactly λ elements.

Properties (2) and (3) are contained in (in fact are equivalent to) the statement (6.5), whereas (1) and (4) can together be expressed as

$$AA' = (k-\lambda)I + \lambda J \tag{5.6}$$

It will be shown that (2) and (3) together imply (1) and (4), and, conversely, (1) and (4) together will imply (2) and (3). Once this has been established, it will follow that A is the incidence matrix of a (v, k, λ)-configuration if A satisfies either (5.5) or (5.6).

Theorem 5.3:

If A is a square (0,1) matrix (i.e. a matrix all of whose entries are 0 or 1) and if A satisfies (5.5) with $k > \lambda$, then (5.6) also holds.

Proof:

Since the diagonal elements of $A'A$ are all k, each column of A contains exactly k 1s. Thus

$$JA = kJ$$

and, on transposing,

$$A'J = kJ$$

Now

$$\left(A' - \sqrt{\left(\frac{\lambda}{v}\right)}J\right)\left(A + \sqrt{\left(\frac{\lambda}{v}\right)}J\right) = A'A + \sqrt{\left(\frac{\lambda}{v}\right)(A'J - JA) - \frac{\lambda}{v}J^2}$$

$$= A'A + \sqrt{\left(\frac{\lambda}{v}\right)(kJ - kJ) - \frac{\lambda}{v}J^2}$$

$$= A'A - \lambda J$$

$$= (k-\lambda)I + \lambda J - \lambda J$$

$$= (k-\lambda)I \text{.........}(5.7)$$

Now if two square matrices M, N are such that $MN = \alpha I$ for some $\alpha \neq 0$, it follows that $\frac{1}{\alpha}M$ and N are inverses of one another and hence commute. Thus $NM = \alpha I$. It now follows from (6.7) that

$$\left(A+\sqrt{\left(\frac{\lambda}{v}\right)}J\right)\left(A'-\sqrt{\left(\frac{\lambda}{v}\right)}J\right)=(k-\lambda)I$$

i.e.

$$AA'+\sqrt{\left(\frac{\lambda}{v}\right)}(JA'-AJ)-\frac{\lambda}{v}J^2=(k-\lambda)I$$

i.e.

$$AA'-(k-\lambda)I-\lambda J=\sqrt{\left(\frac{\lambda}{v}\right)(AJ-JA')}$$

Denote the left-hand side by H, and the right-hand side by K, and note that the matrix H is symmetric ($H'=H$) whereas K is skew symmetric ($K'=-K$). Since $H=K$, it follows that $H=H'=K'=-K=-H$, so that $H=0$. Thus finally

$$AA'-(k-\lambda)I-\lambda J=0$$

i.e.

$$AA'=(k-\lambda)I+\lambda J$$

as required.

This proof is due to I. S. Murphy [22].

Example 5.2:

A finite projective plane of order n is defined to be a (v,k,λ)-configuration for which $v=n^2+n+1, k=n+1$, and $\lambda=1$, for some positive integer $n\geqslant 2$. The seven-point plane corresponds to $n=2$. In a plane of order n there are therefore (n^2+n+1) points and (n^2+n+1) lines, and the four properties listed on p. 83 become as follows.

(1) Any line contains ($n+1$) points.

(2) Any point lies on $(n+1)$ lines.

(3) Any pair of points are joined oy exactly one line.

(4) Any pair of lines intersect in exactly one point.

These four properties can be checked for $n = 2$ by studying Fig. 5.1. The next plane, corresponding to $n = 3$, is a thirteen-point plane with 4 points on each line and 4 lines through each point. See the Exercises 5.4 for its construction.

The major unsolved problem for finite projective planes is to find all those values of n for which a plane of order n exists. The following statements sum up the state of present knowledge.

(a) A plane of order n definitely exists if $n \geqslant 2$ is a prime or a power of a prime.

(b) No plane of any other order is known to exist.

(c) There is definitely no plane of order 6 , or in general of any order n, where n is of the form ($4k + 1$) or ($4k + 2$), and is divisible an odd number of times by a prime of the form ($4h + 3$).

The smallest values of n which are excluded by (c) are $n = 6{,}14{,}22$. The smallest number not covered by (a) and (c) is 10, and it is still not known whether or not a plane of order 10 exists:

Unsolved problem: Is it possible to construct a square (0,1)-matrix A with 111 rows and 111 columns, each row and column containing exactly eleven 1s, such that

$$AA' = 10I + J.$$

The statement (c) above is due to two North American mathematicians Bruck and Ryser. Their proof is a delightful example of the ingenuity and cunning which abound in this branch of mathematics. The proof is accessible to anyone who has done a little matrix algebra, and is now presented in the simplest case of $n = 6$.

Theorem 5.4.

There is no finite projective plane of order 6. It will be convenient to note a few preliminary results before embarking on the proof of the theorem. In what follows, I_n will denote the $n \times n$ unit matrix.

Lemma 1. If H is the 4×4 matrix defined by

$$H = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

then $HH' = 6I_4$.

Lemma 2.

There are no integers a, b, c such that $a^2 + b^2 = 6c^2$,

apart from $a = b = c = 0$.

Proof of Lemma 2.

Suppose such integers do exist. If a, b, c have a common factor it can be divided out, so it can be assumed that no positive integer > 1 divides each of a, b, c. Now $6c^2$ is divisible by 3, so $a^2 + b^2$ must also be divisible by 3. The reader should be able to check that a sum $a^2 + b^2$ can only be divisible by 3 if both a and b are divisible by 3. But then a^2 and b^2, and hence $a^2 + b^2$, are divisible by 9. This implies that $6c^2$ must be divisible by 9, i.e. $2c^2$ is divisible by 3. Thus 3 divides c^2 and nence also divides c. But to have a, b, c all divisible by 3 is a contradiction.

Proof of Theorem 5.4. Suppose there is a plane of order 6. Its incidence matrix A will have 43 rows and columns, and will satisfy

$$AA' = \begin{bmatrix} 7 & 1 & 1 & \dots & 1 \\ 1 & 7 & 1 & \dots & 1 \\ \vdots & & & & \\ 1 & 1 & 1 & \dots & 7 \end{bmatrix}$$

Thus, if

$$B = \begin{bmatrix} & A & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{44\times 44}$$

then

$$BI_{44}B' = \begin{bmatrix} 7 & 1 & \dots & 1 & 0 \\ 1 & 7 & & 1 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 7 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{5.8}$$

Also; if H is as in Lemma 1, and if

$$K = \begin{bmatrix} H & & \\ H & & 0 \\ 0^{H} & & \\ & & H \end{bmatrix}_{44\times 44}$$

then

$$KI_{44}K' = 6I_{44}. \tag{6.9}$$

Now the quadratic form associated with the matrix on the right of (6.8) is

$$7(x_1^2 + \dots + x_{43}^2) + x_{44}^2 + \sum_{\substack{i\neq j \\ 1 \leqslant i,j \leqslant 43}} x_i x_j$$

$$= (x_1 + \dots + x_{43})^2 + x_{44}^2 + 6(x_1^2 + \dots + x_{43}^2)$$

and the quadratic form associated with the matrix on the right of (5.9) is $6(x_1^2 + \dots + x_{44}^2)$ By (5.8) and (5.9), these forms are both transformed into the form associated with the matrix I_{44} by a non-singular linear change of variable. Combining these changes together, a non-singular change of variable

$$\begin{bmatrix} x_1 \\ \dots \\ x_{44} \end{bmatrix} = P \begin{bmatrix} y_1 \\ \dots \\ y_{44} \end{bmatrix}$$

Must exist such that

$$6(y_1^2 + \dots + y_{44}^2) = (x_1 + \dots + x_{43})^2 + x_{44}^2 + 6(x_1^2 + \dots + x_{43}^2) \dots\dots\dots (5.10)$$

the matrix P being non-singular and with rational numbers as its entries. In particular, there are rational $p_{1,1} \dots \dots . p_{1,44}$ such that

$x_1 = p_{1,1}y_1 + \cdots \dots . p_{1,44}y_{44} \dots \dots \dots$(5.11)

If $p_{1,1} \neq 1$, put $x_1 = y_1$.If $p_{1,1}$= 1, put $x_1 = -y_1$,. In either case, $x_1^2 = y_1^2$ and, by (5.11), withx_1, replaced by y_1 , y_1 now depends on y_2,---, y_{44}. In the relation for x_2 corresponding to (5.11), y_1 can there- fore be replaced to give

$x_2 = q_2y_2 + \cdots \dots . q_{44}y_{44}$,

with each q_{i} rational. Now set q_2 =$\pm$ y_2 as before. This induces a dependence relation expressing y in terms of y3, . . ., y_{44}. Continue this process to reach eventually

$x_{43} = r_{43}y_{43} + r_{44}y_{44}$

Now put $x_{43} = \pm y_{43}$ to get $y_{43} = \pm gy_{44}$

for some rational number g. So far, the y_i have been unspecified. Choose y_{44} to be any non-zero rational. Then y_{43}, . . ., y_1 are all uniquely specified, as are all the xx_i;. Moreover, x_{i}^2 =y_{i}^2 for eachi=1,..., 43, so (6.10) becomes

$6y_{44}^2 = y_{44}^2 + (x_2 + \cdots \dots . x_{44})^2$

Thus, there is a rational solution of $6c^2 = a^2 + b^2$. On multiplying through- out by the square of the denominator, a contradiction to Lemma 2 is obtained. Using ideas similar to those of the above proof, it is possible to obtain a much more general result which rules out the existence of some potential symmetric (v, k, λ) configurations where λ(v — 1) = k (k — 1) holds.

Example 5.3:

(a) There is no symmetric (46, 10, 2) design since 10 — 2= 8 is not a square.

(b) There is no symmetric (29, 8, 2) design since the equation $z^2 = 6x^2 + 2y^2$ has no non-trivial integer solution. (Imitate the proof of Lemma 2: z must be even, so

$z = 2w$, so $2w^2 = 3x^2 + y^2$, so $y^2 + w^2$ must be divisible by 3; so y and w are themselves divisible by 3, and hence so is x.)

The special case $\lambda = 1$, $k = n+1$, $v = n^2 + n + 1$ has v odd, and the equation which has to have a nontrivial solution is $z^2 = nx^2$ (= 1) $n(n+1)/2\ y^2$. If n is of the form 4m+1 or 4m + 2 then $n(n + 1)/2$ is odd and so the equation becomes $y^2 + z^2 = nx^2$; standard number theory shows that this requires n to be of the form described earlier. If n is of the form 4m or 4m + 3, then ½ n(n + 1) is even and the equation becomes $z^2 — y^2 = nx^2$, this does have non-trivial solutions: (x, y, z)= (1, 2m + 1, 2m + 2) if n=4m + 3,(x, y, z)=(1,m— 1,m+1)if n=4m.

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